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Duality Based on the Galois Connection. Part I

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Summary. In the paper, we investigate the duality of categories of complete lattices and maps preserving suprema or infima according to [12, p. 179–183; 1.1–1.12]. The duality is based on the concept of the Galois connection.

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The papers [20], [8], [19], [21], [9], [16], [1], [23], [17], [25], [24], [18], [11], [14], [27], [22], [13], [3], [10], [4], [15], [7], [6], [2], [26], and [5] provide the terminology and notation for this paper.

1. Infs-preserving and Sups-preserving Maps

Let S, T be complete lattices. One can check that there exists a connection between S and T which is Galois.

Next we state the proposition

(1) Let S, T, S', T' be non empty relational structures. Suppose that

(i) the relational structure of S = the relational structure of S', and

(ii) the relational structure of T = the relational structure of T'.

Let c be a connection between S and T and c' be a connection between S' and T'. If $c = c'$, then if c is Galois, then c' is Galois.

Let S, T be lattices and let g be a map from S into T. Let us assume that S is complete and T is complete and g is infs-preserving. The lower adjoint of g is a map from T into S and is defined as follows:

(Def. 1) $\langle g$, the lower adjoint of $g\rangle$ is Galois.

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Let S, T be lattices and let d be a map from T into S. Let us assume that S is complete and T is complete and d is sups-preserving. The upper adjoint of d is a map from S into T and is defined as follows:

(Def. 2) $\langle$the upper adjoint of d, $d\rangle$ is Galois.

Let S, T be complete lattices and let g be an infs-preserving map from S into T. One can verify that the lower adjoint of g is lower adjoint.

Let S, T be complete lattices and let d be a sups-preserving map from T into S. One can check that the upper adjoint of d is upper adjoint.

The following two propositions are true:

(2) Let S, T be complete lattices, g be an infs-preserving map from S into T, and t be an element of T. Then (the lower adjoint of g)$(t) = \inf(g^{-1}(\uparrow t))$.

(3) Let S, T be complete lattices, d be a sups-preserving map from T into S, and s be an element of S. Then (the upper adjoint of d)$(s) = \sup(d^{-1}(\downarrow s))$.

Let S, T be relational structures and let f be a function from the carrier of S into the carrier of T. The functor $f^{\rm op}$ yielding a map from $S^{\rm op}$ into $T^{\rm op}$ is defined as follows:

(Def. 3) $f^{\rm op} = f$.

Let S, T be complete lattices and let g be an infs-preserving map from S into T. One can verify that $g^{\rm op}$ is lower adjoint.

Let S, T be complete lattices and let d be a sups-preserving map from S into T. Observe that $d^{\rm op}$ is upper adjoint.

We now state several propositions:

(4) Let S, T be complete lattices and g be an infs-preserving map from S into T. Then the lower adjoint of $g =$ the upper adjoint of $g^{\rm op}$.

(5) Let S, T be complete lattices and d be a sups-preserving map from S into T. Then the lower adjoint of $d^{\rm op} =$ the upper adjoint of d.

(6) For every non empty relational structure L holds $\langle {\rm id}_L, {\rm id}_L\rangle$ is Galois.

(7) For every complete lattice L holds the lower adjoint of ${\rm id}_L = {\rm id}_L$ and the upper adjoint of ${\rm id}_L = {\rm id}_L$.

(8) Let L_1, L_2, L_3 be complete lattices, g_1 be an infs-preserving map from L_1 into L_2, and g_2 be an infs-preserving map from L_2 into L_3. Then the lower adjoint of $g_2 \cdot g_1 =$ (the lower adjoint of g_1) $\cdot$ (the lower adjoint of g_2).

(9) Let L_1, L_2, L_3 be complete lattices, d_1 be a sups-preserving map from L_1 into L_2, and d_2 be a sups-preserving map from L_2 into L_3. Then the upper adjoint of $d_2 \cdot d_1 =$ (the upper adjoint of d_1) $\cdot$ (the upper adjoint of d_2).

(10) Let S, T be complete lattices and g be an infs-preserving map from S into T. Then the upper adjoint of the lower adjoint of $g = g$.

(11) Let S, T be complete lattices and d be a sups-preserving map from S into T. Then the lower adjoint of the upper adjoint of $d = d$.

(12) Let C be a non empty category structure and a, b, f be sets. Suppose $f \in$ (the arrows of C)(a, b). Then there exist objects o_1, o_2 of C such that $o_1 = a$ and $o_2 = b$ and $f \in \langle o_1, o_2 \rangle$ and f is a morphism from o_1 to o_2.

Let W be a non empty set. Let us assume that there exists an element w of W such that w is non empty. The functor INF_W yields a lattice-wise strict category and is defined by the conditions (Def. 4).

(Def. 4)(i) For every lattice x holds x is an object of INF_W iff x is strict and complete and the carrier of $x \in W$, and

(ii) for all objects a, b of INF_W and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff f is infs-preserving.

Let W be a non empty set. Let us assume that there exists an element w of W such that w is non empty. The functor SUP_W yields a lattice-wise strict category and is defined by the conditions (Def. 5).

(Def. 5)(i) For every lattice x holds x is an object of SUP_W iff x is strict and complete and the carrier of $x \in W$, and

(ii) for all objects a, b of SUP_W and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff f is sups-preserving.

Let W be a set with a non-empty element. Observe that INF_W has complete lattices and SUP_W has complete lattices.

One can prove the following propositions:

(13) Let W be a set with a non-empty element and L be a lattice. Then L is an object of INF_W if and only if L is strict and complete and the carrier of $L \in W$.

(14) Let W be a set with a non-empty element, a, b be objects of INF_W, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is an infs-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

(15) Let W be a set with a non-empty element and L be a lattice. Then L is an object of SUP_W if and only if L is strict and complete and the carrier of $L \in W$.

(16) Let W be a set with a non-empty element, a, b be objects of SUP_W, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is a sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

(17) For every set W with a non-empty element holds the carrier of $INF_W =$ the carrier of SUP_W.

Let W be a set with a non-empty element. The functor $\mathrm{LowerAdj}_W$ yields a contravariant strict functor from INF_W to SUP_W and is defined by the conditions (Def. 6).

(Def. 6)(i) For every object a of INF_W holds $\mathrm{LowerAdj}_W(a) = \mathbb{L}_a$, and

(ii) for all objects a, b of INF_W such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathrm{LowerAdj}_W(f) =$ the lower adjoint of ${}^{@}f$.

The functor $\mathrm{UpperAdj}_W$ yields a contravariant strict functor from SUP_W to INF_W and is defined by the conditions (Def. 7).

(Def. 7)(i) For every object a of SUP_W holds $\mathrm{UpperAdj}_W(a) = \mathbb{L}_a$, and

(ii) for all objects a, b of SUP_W such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathrm{UpperAdj}_W(f) =$ the upper adjoint of ${}^{@}f$.

Let W be a set with a non-empty element. Observe that $\mathrm{LowerAdj}_W$ is bijective and $\mathrm{UpperAdj}_W$ is bijective.

We now state several propositions:

(18) For every set W with a non-empty element holds $(\mathrm{LowerAdj}_W)^{-1} = \mathrm{UpperAdj}_W$ and $(\mathrm{UpperAdj}_W)^{-1} = \mathrm{LowerAdj}_W$.

(19) For every set W with a non-empty element holds $\mathrm{LowerAdj}_W \cdot \mathrm{UpperAdj}_W = \mathrm{id}_{SUP_W}$ and $\mathrm{UpperAdj}_W \cdot \mathrm{LowerAdj}_W = \mathrm{id}_{INF_W}$.

(20) For every set W with a non-empty element holds INF_W, SUP_W are anti-isomorphic.

(21) For every set W with a non-empty element holds INF_W and SUP_W are anti-isomorphic under $\mathrm{LowerAdj}_W$.

(22) For every set W with a non-empty element holds SUP_W and INF_W are anti-isomorphic under $\mathrm{UpperAdj}_W$.

2. Scott Continuous Maps and Continuous Lattices

Next we state the proposition

(23) Let S, T be complete lattices and g be an infs-preserving map from S into T. Then g is directed-sups-preserving if and only if for every Scott topological augmentation X of T and for every Scott topological augmentation Y of S and for every open subset V of X holds $\uparrow$((the lower adjoint of $g)^{\circ}V$) is an open subset of Y.

Let S, T be non empty reflexive relational structures and let f be a map from S into T. We say that f is waybelow-preserving if and only if:

(Def. 8) For all elements x, y of S such that $x \ll y$ holds $f(x) \ll f(y)$.

We now state two propositions:

(24) Let S, T be complete lattices and g be an infs-preserving map from S into T. Suppose g is directed-sups-preserving. Then the lower adjoint of g is waybelow-preserving.

(25) Let S be a complete lattice, T be a complete continuous lattice, and g be an infs-preserving map from S into T. Suppose the lower adjoint of g is waybelow-preserving. Then g is directed-sups-preserving.

Let S, T be topological spaces and let f be a map from S into T. We say that f is relatively open if and only if:

(Def. 9) For every open subset V of S holds $f^\circ V$ is an open subset of $T \restriction \operatorname{rng} f$.

One can prove the following propositions:

(26) Let X, Y be non empty topological spaces and d be a map from X into Y. Then d is relatively open if and only if d° is open.

(27) Let S, T be complete lattices, g be an infs-preserving map from S into T, X be a Scott topological augmentation of T, Y be a Scott topological augmentation of S, and V be an open subset of X. Then (the lower adjoint of $g)^\circ V = \operatorname{rng}$ (the lower adjoint of $g) \cap \uparrow(($the lower adjoint of $g)^\circ V)$.

(28) Let S, T be complete lattices, g be an infs-preserving map from S into T, X be a Scott topological augmentation of T, and Y be a Scott topological augmentation of S. Suppose that for every open subset V of X holds $\uparrow(($the lower adjoint of $g)^\circ V)$ is an open subset of Y. Let d be a map from X into Y. If $d =$ the lower adjoint of g, then d is relatively open.

Let X, Y be complete lattices and let f be a sups-preserving map from X into Y. One can check that $\operatorname{Im} f$ is complete.

Next we state four propositions:

(29) Let S, T be complete lattices, g be an infs-preserving map from S into T, X be a Scott topological augmentation of T, Y be a Scott topological augmentation of S, Z be a Scott topological augmentation of Im (the lower adjoint of g), d be a map from X into Y, and d' be a map from X into Z. Suppose $d =$ the lower adjoint of g and $d' = d$. If d is relatively open, then d' is open.

(30) Let T_1, T_2, S_1, S_2 be topological structures. Suppose that
 (i) the topological structure of $T_1 =$ the topological structure of T_2, and
 (ii) the topological structure of $S_1 =$ the topological structure of S_2.
 If S_1 is a subspace of T_1, then S_2 is a subspace of T_2.

(31) For every topological structure T holds $T \restriction \Omega_T =$ the topological structure of T.

(32) Let S, T be complete lattices and g be an infs-preserving map from S into T. Suppose g is one-to-one. Let X be a Scott topological augmentation of T, Y be a Scott topological augmentation of S, and d be a map from X into Y. Suppose $d =$ the lower adjoint of g. Then g is directed-sups-preserving if and only if d is open.

Let X be a complete lattice and let f be a projection map from X into X. One can verify that $\operatorname{Im} f$ is complete.

We now state a number of propositions:

(33) Let L be a complete lattice and k be a kernel map from L into L. Then
 (i) k° is infs-preserving,

(ii) $k_\circ$ is sups-preserving,
(iii) the lower adjoint of $k^\circ = k_\circ$, and
(iv) the upper adjoint of $k_\circ = k^\circ$.

(34) Let L be a complete lattice and k be a kernel map from L into L. Then k is directed-sups-preserving if and only if k° is directed-sups-preserving.

(35) Let L be a complete lattice and k be a kernel map from L into L. Then k is directed-sups-preserving if and only if for every Scott topological augmentation X of $\operatorname{Im} k$ and for every Scott topological augmentation Y of L and for every subset V of L such that V is an open subset of X holds $\uparrow V$ is an open subset of Y.

(36) Let L be a complete lattice, S be a sups-inheriting non empty full relational substructure of L, x, y be elements of L, and a, b be elements of S. If $a = x$ and $b = y$, then if $x \ll y$, then $a \ll b$.

(37) Let L be a complete lattice and k be a kernel map from L into L. Suppose k is directed-sups-preserving. Let x, y be elements of L and a, b be elements of $\operatorname{Im} k$. If $a = x$ and $b = y$, then $x \ll y$ iff $a \ll b$.

(38) Let L be a complete lattice and k be a kernel map from L into L. Suppose that

(i) $\operatorname{Im} k$ is continuous, and
(ii) for all elements x, y of L and for all elements a, b of $\operatorname{Im} k$ such that $a = x$ and $b = y$ holds $x \ll y$ iff $a \ll b$.

Then k is directed-sups-preserving.

(39) Let L be a complete lattice and c be a closure map from L into L. Then
(i) c° is sups-preserving,
(ii) $c_\circ$ is infs-preserving,
(iii) the upper adjoint of $c^\circ = c_\circ$, and
(iv) the lower adjoint of $c_\circ = c^\circ$.

(40) Let L be a complete lattice and c be a closure map from L into L. Then $\operatorname{Im} c$ is directed-sups-inheriting if and only if $c_\circ$ is directed-sups-preserving.

(41) Let L be a complete lattice and c be a closure map from L into L. Then $\operatorname{Im} c$ is directed-sups-inheriting if and only if for every Scott topological augmentation X of $\operatorname{Im} c$ and for every Scott topological augmentation Y of L and for every map f from Y into X such that $f = c$ holds f is open.

(42) Let L be a complete lattice and c be a closure map from L into L. If $\operatorname{Im} c$ is directed-sups-inheriting, then c° is waybelow-preserving.

(43) Let L be a continuous complete lattice and c be a closure map from L into L. If c° is waybelow-preserving, then $\operatorname{Im} c$ is directed-sups-inheriting.

3. Duality of Subcategories of *INF* and *SUP*

Let W be a non empty set. The functor $INF^{\uparrow}_W$ yielding a strict non empty subcategory of INF_W is defined by the conditions (Def. 10).

(Def. 10)(i) Every object of INF_W is an object of $INF^{\uparrow}_W$, and

(ii) for all objects a, b of INF_W and for all objects a', b' of $INF^{\uparrow}_W$ such that $a' = a$ and $b' = b$ and $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $f \in \langle a', b'\rangle$ iff ${}^{@}f$ is directed-sups-preserving.

Let W be a set with a non-empty element. The functor SUP^0_W yields a strict non empty subcategory of SUP_W and is defined by the conditions (Def. 11).

(Def. 11)(i) Every object of SUP_W is an object of SUP^0_W, and

(ii) for all objects a, b of SUP_W and for all objects a', b' of SUP^0_W such that $a' = a$ and $b' = b$ and $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $f \in \langle a', b'\rangle$ iff the upper adjoint of ${}^{@}f$ is directed-sups-preserving.

The following propositions are true:

(44) Let S be a non empty relational structure, T be a non empty reflexive antisymmetric relational structure, t be an element of T, and X be a non empty subset of S. Then $S \longmapsto t$ preserves sup of X and $S \longmapsto t$ preserves inf of X.

(45) Let S be a non empty relational structure and T be a lower-bounded non empty reflexive antisymmetric relational structure. Then $S \longmapsto \bot_T$ is sups-preserving.

(46) Let S be a non empty relational structure and T be an upper-bounded non empty reflexive antisymmetric relational structure. Then $S \longmapsto \top_T$ is infs-preserving.

Let S be a non empty relational structure and let T be an upper-bounded non empty reflexive antisymmetric relational structure. Observe that $S \longmapsto \top_T$ is directed-sups-preserving and infs-preserving.

Let S be a non empty relational structure and let T be a lower-bounded non empty reflexive antisymmetric relational structure. Observe that $S \longmapsto \bot_T$ is filtered-infs-preserving and sups-preserving.

Let S be a non empty relational structure and let T be an upper-bounded non empty reflexive antisymmetric relational structure. Note that there exists a map from S into T which is directed-sups-preserving and infs-preserving.

Let S be a non empty relational structure and let T be a lower-bounded non empty reflexive antisymmetric relational structure. One can check that there exists a map from S into T which is filtered-infs-preserving and sups-preserving.

Next we state several propositions:

(47) Let W be a set with a non-empty element and L be a lattice. Then L is an object of $\mathit{INF}^{\uparrow}_{W}$ if and only if L is strict and complete and the carrier of $L \in W$.

(48) Let W be a set with a non-empty element, a, b be objects of $\mathit{INF}^{\uparrow}_{W}$, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is a directed-sups-preserving infs-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

(49) Let W be a set with a non-empty element and L be a lattice. Then L is an object of SUP^{0}_{W} if and only if L is strict and complete and the carrier of $L \in W$.

(50) Let W be a set with a non-empty element, a, b be objects of SUP^{0}_{W}, and f be a set. Then $f \in \langle a, b \rangle$ if and only if there exists a sups-preserving map g from $\mathbb{L}_a$ into $\mathbb{L}_b$ such that $g = f$ and the upper adjoint of g is directed-sups-preserving.

(51) For every set W with a non-empty element holds $\mathit{INF}^{\uparrow}_{W} = \mathrm{Intersect}(\mathit{INF}_{W}, \mathit{UPS}_{W})$.

Let W be a set with a non-empty element. The functor CL_{W} yielding a strict full non empty subcategory of $\mathit{INF}^{\uparrow}_{W}$ is defined as follows:

(Def. 12) For every object a of $\mathit{INF}^{\uparrow}_{W}$ holds a is an object of CL_{W} iff $\mathbb{L}_a$ is continuous.

Let W be a set with a non-empty element. Observe that CL_{W} has complete lattices.

One can prove the following two propositions:

(52) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of CL_{W} if and only if L is strict, complete, and continuous.

(53) Let W be a set with a non-empty element, a, b be objects of CL_{W}, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is an infs-preserving directed-sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

Let W be a set with a non-empty element. The functor $\mathit{CL}^{\mathrm{op}}_{W}$ yields a strict full non empty subcategory of SUP^{0}_{W} and is defined by:

(Def. 13) For every object a of SUP^{0}_{W} holds a is an object of $\mathit{CL}^{\mathrm{op}}_{W}$ iff $\mathbb{L}_a$ is continuous.

Next we state several propositions:

(54) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of $\mathit{CL}^{\mathrm{op}}_{W}$ if and only if L is strict, complete, and continuous.

(55) Let W be a set with a non-empty element, a, b be objects of $\mathit{CL}^{\mathrm{op}}_{W}$, and f be a set. Then $f \in \langle a, b \rangle$ if and only if there exists a sups-preserving map g from $\mathbb{L}_a$ into $\mathbb{L}_b$ such that $g = f$ and the upper adjoint of g is directed-sups-preserving.

(56) For every set W with a non-empty element holds $INF^{\uparrow}_W$ and SUP^0_W are anti-isomorphic under $\mathrm{LowerAdj}_W$.

(57) For every set W with a non-empty element holds SUP^0_W and $INF^{\uparrow}_W$ are anti-isomorphic under $\mathrm{UpperAdj}_W$.

(58) For every set W with a non-empty element holds CL_W and CL^{op}_W are anti-isomorphic under $\mathrm{LowerAdj}_W$.

(59) For every set W with a non-empty element holds CL^{op}_W and CL_W are anti-isomorphic under $\mathrm{UpperAdj}_W$.

4. Compact Preserving Maps and Sup-semilattices Morphisms

Let S, T be non empty reflexive relational structures and let f be a map from S into T. We say that f is compact-preserving if and only if:

(Def. 14) For every element s of S such that s is compact holds $f(s)$ is compact.

One can prove the following propositions:

(60) Let S, T be complete lattices and d be a sups-preserving map from T into S. If d is waybelow-preserving, then d is compact-preserving.

(61) Let S, T be complete lattices and d be a sups-preserving map from T into S. Suppose T is algebraic and d is compact-preserving. Then d is waybelow-preserving.

(62) Let R, S, T be non empty relational structures, X be a subset of R, f be a map from R into S, and g be a map from S into T. Suppose f preserves sup of X and g preserves sup of $f^{\circ}X$. Then $g \cdot f$ preserves sup of X.

Let S, T be non empty relational structures and let f be a map from S into T. We say that f is finite-sups-preserving if and only if:

(Def. 15) For every finite subset X of S holds f preserves sup of X.

We say that f is bottom-preserving if and only if:

(Def. 16) f preserves sup of $\emptyset_S$.

Next we state the proposition

(63) Let R, S, T be non empty relational structures, f be a map from R into S, and g be a map from S into T. Suppose f is finite-sups-preserving and g is finite-sups-preserving. Then $g \cdot f$ is finite-sups-preserving.

Let S, T be non empty antisymmetric lower-bounded relational structures and let f be a map from S into T. Let us observe that f is bottom-preserving if and only if:

(Def. 17) $f(\bot_S) = \bot_T$.

Let L be a non empty relational structure and let S be a relational substructure of L. We say that S is finite-sups-inheriting if and only if:

(Def. 18) For every finite subset X of S such that sup X exists in L holds $\bigsqcup_L X \in$ the carrier of S.

We say that S is bottom-inheriting if and only if:

(Def. 19) $\bot_L \in$ the carrier of S.

Let S, T be non empty relational structures. Observe that every map from S into T which is sups-preserving is also bottom-preserving.

Let L be a lower-bounded antisymmetric non empty relational structure. Note that every relational substructure of L which is finite-sups-inheriting is also bottom-inheriting and join-inheriting.

Let L be a non empty relational structure. One can check that every relational substructure of L which is sups-inheriting is also finite-sups-inheriting.

Let S, T be lower-bounded non empty posets. One can verify that there exists a map from S into T which is sups-preserving.

Let L be a lower-bounded antisymmetric non empty relational structure. Observe that every full relational substructure of L which is bottom-inheriting is also non empty and lower-bounded.

Let L be a lower-bounded antisymmetric non empty relational structure. Note that there exists a relational substructure of L which is non empty, sups-inheriting, finite-sups-inheriting, bottom-inheriting, and full.

Next we state the proposition

(64) Let L be a lower-bounded antisymmetric non empty relational structure and S be a non empty bottom-inheriting full relational substructure of L. Then $\bot_S = \bot_L$.

Let L be a lower-bounded non empty poset with l.u.b.'s. Note that every full relational substructure of L which is bottom-inheriting and join-inheriting is also finite-sups-inheriting.

Next we state two propositions:

(65) Let S, T be non empty relational structures and f be a map from S into T. Suppose f is finite-sups-preserving. Then f is join-preserving and bottom-preserving.

(66) Let S, T be lower-bounded posets with l.u.b.'s and f be a map from S into T. Suppose f is join-preserving and bottom-preserving. Then f is finite-sups-preserving.

Let S, T be non empty relational structures. One can check that every map from S into T which is sups-preserving is also finite-sups-preserving and every map from S into T which is finite-sups-preserving is also join-preserving and bottom-preserving.

Let S be a non empty relational structure and let T be a lower-bounded non empty reflexive antisymmetric relational structure. Observe that there exists a map from S into T which is sups-preserving and finite-sups-preserving.

Let L be a lower-bounded non empty poset. One can check that CompactSublatt(L) is lower-bounded.

One can prove the following propositions:

(67) Let S be a relational structure, T be a non empty relational structure, f be a map from S into T, S' be a relational substructure of S, and T' be a relational substructure of T. Suppose f°(the carrier of S') $\subseteq$ the carrier of T'. Then $f\restriction$the carrier of S' is a map from S' into T'.

(68) Let S, T be lattices, f be a join-preserving map from S into T, S' be a non empty join-inheriting full relational substructure of S, T' be a non empty join-inheriting full relational substructure of T, and g be a map from S' into T'. If $g = f\restriction$the carrier of S', then g is join-preserving.

(69) Let S, T be lower-bounded lattices, f be a finite-sups-preserving map from S into T, S' be a non empty finite-sups-inheriting full relational substructure of S, T' be a non empty finite-sups-inheriting full relational substructure of T, and g be a map from S' into T'. If $g = f\restriction$the carrier of S', then g is finite-sups-preserving.

Let L be a complete lattice. One can verify that CompactSublatt(L) is finite-sups-inheriting.

Next we state two propositions:

(70) Let S, T be complete lattices and d be a sups-preserving map from T into S. Then d is compact-preserving if and only if $d\restriction$the carrier of CompactSublatt(T) is a finite-sups-preserving map from CompactSublatt(T) into CompactSublatt(S).

(71) Let S, T be complete lattices. Suppose T is algebraic. Let g be an infs-preserving map from S into T. Then g is directed-sups-preserving if and only if (the lower adjoint of g)$\restriction$the carrier of CompactSublatt(T) is a finite-sups-preserving map from CompactSublatt(T) into CompactSublatt(S).

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