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# Miscellaneous Facts about Functors

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**Summary.** In the paper we show useful facts concerning reverse and inclusion functors and the restriction of functors. We also introduce a new notation for the intersection of categories and the isomorphism under arbitrary functors.

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The notation and terminology used in this paper have been introduced in the following articles: [11], [12], [15], [13], [7], [2], [3], [4], [9], [14], [5], [10], [16], [17], [8], [1], and [6].

## 1. Reverse Functors

The following propositions are true:

(1) Let $A$, $B$ be transitive non empty category structures with units and $F$ be a feasible reflexive functor structure from $A$ to $B$. Suppose $F$ is coreflexive and bijective. Let $a$ be an object of $A$ and $b$ be an object of $B$. Then $F(a) = b$ if and only if $F^{-1}(b) = a$.

(2) Let $A$, $B$ be transitive non empty category structures with units, $F$ be a precovariant feasible functor structure from $A$ to $B$, and $G$ be a precovariant feasible functor structure from $B$ to $A$. Suppose $F$ is bijective and $G = F^{-1}$. Let $a_1$, $a_2$ be objects of $A$. Suppose $\langle a_1, a_2 \rangle \neq \emptyset$. Let $f$ be a morphism from $a_1$ to $a_2$ and $g$ be a morphism from $F(a_1)$ to $F(a_2)$. Then $F(f) = g$ if and only if $G(g) = f$.

(3) Let $A$, $B$ be transitive non empty category structures with units, $F$ be a precontravariant feasible functor structure from $A$ to $B$, and $G$ be




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a precontravariant feasible functor structure from $B$ to $A$. Suppose $F$ is bijective and $G = F^{-1}$. Let $a_1$, $a_2$ be objects of $A$. Suppose $\langle a_1, a_2 \rangle \neq \emptyset$. Let $f$ be a morphism from $a_1$ to $a_2$ and $g$ be a morphism from $F(a_2)$ to $F(a_1)$. Then $F(f) = g$ if and only if $G(g) = f$.

(4) Let $A$, $B$ be categories and $F$ be a functor from $A$ to $B$. Suppose $F$ is bijective. Let $G$ be a functor from $B$ to $A$. If $F \cdot G = \mathrm{id}_B$, then the functor structure of $G = F^{-1}$.

(5) Let $A$, $B$ be categories and $F$ be a functor from $A$ to $B$. Suppose $F$ is bijective. Let $G$ be a functor from $B$ to $A$. If $G \cdot F = \mathrm{id}_A$, then the functor structure of $G = F^{-1}$.

(6) Let $A$, $B$ be categories and $F$ be a covariant functor from $A$ to $B$. Suppose $F$ is bijective. Let $G$ be a covariant functor from $B$ to $A$. Suppose that

(i) for every object $b$ of $B$ holds $F(G(b)) = b$, and

(ii) for all objects $a$, $b$ of $B$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism $f$ from $a$ to $b$ holds $F(G(f)) = f$.

Then the functor structure of $G = F^{-1}$.

(7) Let $A$, $B$ be categories and $F$ be a contravariant functor from $A$ to $B$. Suppose $F$ is bijective. Let $G$ be a contravariant functor from $B$ to $A$. Suppose that

(i) for every object $b$ of $B$ holds $F(G(b)) = b$, and

(ii) for all objects $a$, $b$ of $B$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism $f$ from $a$ to $b$ holds $F(G(f)) = f$.

Then the functor structure of $G = F^{-1}$.

(8) Let $A$, $B$ be categories and $F$ be a covariant functor from $A$ to $B$. Suppose $F$ is bijective. Let $G$ be a covariant functor from $B$ to $A$. Suppose that

(i) for every object $a$ of $A$ holds $G(F(a)) = a$, and

(ii) for all objects $a$, $b$ of $A$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism $f$ from $a$ to $b$ holds $G(F(f)) = f$.

Then the functor structure of $G = F^{-1}$.

(9) Let $A$, $B$ be categories and $F$ be a contravariant functor from $A$ to $B$. Suppose $F$ is bijective. Let $G$ be a contravariant functor from $B$ to $A$. Suppose that

(i) for every object $a$ of $A$ holds $G(F(a)) = a$, and

(ii) for all objects $a$, $b$ of $A$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism $f$ from $a$ to $b$ holds $G(F(f)) = f$.

Then the functor structure of $G = F^{-1}$.

## 2. Intersection of Categories

Let $A$, $B$ be category structures. We say that $A$ and $B$ have the same composition if and only if:

(Def. 1) For all sets $a_1$, $a_2$, $a_3$ holds (the composition of $A$)($\langle a_1, a_2, a_3 \rangle$) $\approx$ (the composition of $B$)($\langle a_1, a_2, a_3 \rangle$).

Let us note that the predicate $A$ and $B$ have the same composition is symmetric.

Next we state three propositions:

(10) Let $A$, $B$ be category structures. Then $A$ and $B$ have the same composition if and only if for all sets $a_1$, $a_2$, $a_3$, $x$ such that $x \in \mathrm{dom}$ (the composition of $A$)($\langle a_1, a_2, a_3 \rangle$) and $x \in \mathrm{dom}$ (the composition of $B$)($\langle a_1, a_2, a_3 \rangle$) holds (the composition of $A$)($\langle a_1, a_2, a_3 \rangle$)($x$) = (the composition of $B$)($\langle a_1, a_2, a_3 \rangle$)($x$).

(11) Let $A$, $B$ be transitive non empty category structures. Then $A$ and $B$ have the same composition if and only if for all objects $a_1$, $a_2$, $a_3$ of $A$ such that $\langle a_1, a_2 \rangle \neq \emptyset$ and $\langle a_2, a_3 \rangle \neq \emptyset$ and for all objects $b_1$, $b_2$, $b_3$ of $B$ such that $\langle b_1, b_2 \rangle \neq \emptyset$ and $\langle b_2, b_3 \rangle \neq \emptyset$ and $b_1 = a_1$ and $b_2 = a_2$ and $b_3 = a_3$ and for every morphism $f_1$ from $a_1$ to $a_2$ and for every morphism $g_1$ from $b_1$ to $b_2$ such that $g_1 = f_1$ and for every morphism $f_2$ from $a_2$ to $a_3$ and for every morphism $g_2$ from $b_2$ to $b_3$ such that $g_2 = f_2$ holds $f_2 \cdot f_1 = g_2 \cdot g_1$.

(12) For all para-functional semi-functional categories $A$, $B$ holds $A$ and $B$ have the same composition.

Let $f$, $g$ be functions. The functor $\mathrm{Intersect}(f, g)$ yielding a function is defined as follows:

(Def. 2) $\mathrm{dom}\,\mathrm{Intersect}(f, g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$ and for every set $x$ such that $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ holds $(\mathrm{Intersect}(f, g))(x) = f(x) \cap g(x)$.

Let us notice that the functor $\mathrm{Intersect}(f, g)$ is commutative.

One can prove the following propositions:

(13) For every set $I$ and for all many sorted sets $A$, $B$ indexed by $I$ holds $\mathrm{Intersect}(A, B) = A \cap B$.

(14) Let $I$, $J$ be sets, $A$ be a many sorted set indexed by $I$, and $B$ be a many sorted set indexed by $J$. Then $\mathrm{Intersect}(A, B)$ is a many sorted set indexed by $I \cap J$.

(15) Let $I$, $J$ be sets, $A$ be a many sorted set indexed by $I$, $B$ be a function, and $C$ be a many sorted set indexed by $J$. If $C = \mathrm{Intersect}(A, B)$, then $C \mathrel{\dot{\subseteq}} A$.

(16) Let $A_1$, $A_2$, $B_1$, $B_2$ be sets, $f$ be a function from $A_1$ into $A_2$, and $g$ be a function from $B_1$ into $B_2$. If $f \approx g$, then $f \cap g$ is a function from $A_1 \cap B_1$ into $A_2 \cap B_2$.

(17) Let $I_1$, $I_2$ be sets, $A_1$, $B_1$ be many sorted sets indexed by $I_1$, $A_2$, $B_2$ be many sorted sets indexed by $I_2$, and $A$, $B$ be many sorted sets indexed by $I_1 \cap I_2$. Suppose $A = \mathrm{Intersect}(A_1, A_2)$ and $B = \mathrm{Intersect}(B_1, B_2)$. Let $F$ be a many sorted function from $A_1$ into $B_1$ and $G$ be a many sorted function from $A_2$ into $B_2$. Suppose that for every set $x$ such that $x \in \mathrm{dom}\, F$ and $x \in \mathrm{dom}\, G$ holds $F(x) \approx G(x)$. Then $\mathrm{Intersect}(F, G)$ is a many sorted function from $A$ into $B$.

(18) Let $I$, $J$ be sets, $F$ be a many sorted set indexed by $[\!: I, I :\!]$, and $G$ be a many sorted set indexed by $[\!: J, J :\!]$. Then there exists a many sorted set $H$ indexed by $[\!: I \cap J, I \cap J :\!]$ such that $H = \mathrm{Intersect}(F, G)$ and $\mathrm{Intersect}(\{\!| F |\!\}, \{\!| G |\!\}) = \{\!| H |\!\}$.

(19) Let $I$, $J$ be sets, $F_1$, $F_2$ be many sorted sets indexed by $[\!: I, I :\!]$, and $G_1$, $G_2$ be many sorted sets indexed by $[\!: J, J :\!]$. Then there exist many sorted sets $H_1$, $H_2$ indexed by $[\!: I \cap J, I \cap J :\!]$ such that $H_1 = \mathrm{Intersect}(F_1, G_1)$ and $H_2 = \mathrm{Intersect}(F_2, G_2)$ and $\mathrm{Intersect}(\{\!| F_1, F_2 |\!\}, \{\!| G_1, G_2 |\!\}) = \{\!| H_1, H_2 |\!\}$.

Let $A$, $B$ be category structures. Let us assume that $A$ and $B$ have the same composition. The functor $\mathrm{Intersect}(A, B)$ yields a strict category structure and is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of $\mathrm{Intersect}(A, B)$ = (the carrier of $A$) $\cap$ (the carrier of $B$),

(ii) the arrows of $\mathrm{Intersect}(A, B)$ = Intersect(the arrows of $A$, the arrows of $B$), and

(iii) the composition of $\mathrm{Intersect}(A, B)$ = Intersect(the composition of $A$, the composition of $B$).

The following propositions are true:

(20) For all category structures $A$, $B$ such that $A$ and $B$ have the same composition holds $\mathrm{Intersect}(A, B) = \mathrm{Intersect}(B, A)$.

(21) Let $A$, $B$ be category structures. Suppose $A$ and $B$ have the same composition. Then $\mathrm{Intersect}(A, B)$ is a substructure of $A$.

(22) Let $A$, $B$ be category structures. Suppose $A$ and $B$ have the same composition. Let $a_1$, $a_2$ be objects of $A$, $b_1$, $b_2$ be objects of $B$, and $o_1$, $o_2$ be objects of $\mathrm{Intersect}(A, B)$. If $o_1 = a_1$ and $o_1 = b_1$ and $o_2 = a_2$ and $o_2 = b_2$, then $\langle o_1, o_2 \rangle = (\langle a_1, a_2 \rangle) \cap (\langle b_1, b_2 \rangle)$.

(23) Let $A$, $B$ be transitive category structures. If $A$ and $B$ have the same composition, then $\mathrm{Intersect}(A, B)$ is transitive.

(24) Let $A$, $B$ be category structures. Suppose $A$ and $B$ have the same composition. Let $a_1$, $a_2$ be objects of $A$, $b_1$, $b_2$ be objects of $B$, and $o_1$, $o_2$ be objects of $\mathrm{Intersect}(A, B)$. Suppose $o_1 = a_1$ and $o_1 = b_1$ and $o_2 = a_2$ and $o_2 = b_2$ and $\langle a_1, a_2 \rangle \neq \emptyset$ and $\langle b_1, b_2 \rangle \neq \emptyset$. Let $f$ be a morphism from $a_1$ to $a_2$ and $g$ be a morphism from $b_1$ to $b_2$. If $f = g$, then $f \in \langle o_1, o_2 \rangle$.

(25) Let $A$, $B$ be non empty category structures with units. Suppose $A$ and $B$ have the same composition. Let $a$ be an object of $A$, $b$ be an object of $B$, and $o$ be an object of $\mathrm{Intersect}(A, B)$. If $o = a$ and $o = b$ and $\mathrm{id}_a = \mathrm{id}_b$, then $\mathrm{id}_a \in \langle o, o \rangle$.

(26) Let $A$, $B$ be categories. Suppose that

(i) $A$ and $B$ have the same composition,

(ii) $\mathrm{Intersect}(A, B)$ is non empty, and

(iii) for every object $a$ of $A$ and for every object $b$ of $B$ such that $a = b$ holds $\mathrm{id}_a = \mathrm{id}_b$.

Then $\mathrm{Intersect}(A, B)$ is a subcategory of $A$.

## 3. Subcategories

The scheme *SubcategoryUniq* deals with a category $\mathcal{A}$, non empty subcategories $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, a unary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

The category structure of $\mathcal{B}$ = the category structure of $\mathcal{C}$

provided the following requirements are met:

- For every object $a$ of $\mathcal{A}$ holds $a$ is an object of $\mathcal{B}$ iff $\mathcal{P}[a]$,
- Let $a$, $b$ be objects of $\mathcal{A}$ and $a'$, $b'$ be objects of $\mathcal{B}$. Suppose $a' = a$ and $b' = b$ and $\langle a, b \rangle \neq \emptyset$. Let $f$ be a morphism from $a$ to $b$. Then $f \in \langle a', b' \rangle$ if and only if $\mathcal{Q}[a, b, f]$,
- For every object $a$ of $\mathcal{A}$ holds $a$ is an object of $\mathcal{C}$ iff $\mathcal{P}[a]$, and
- Let $a$, $b$ be objects of $\mathcal{A}$ and $a'$, $b'$ be objects of $\mathcal{C}$. Suppose $a' = a$ and $b' = b$ and $\langle a, b \rangle \neq \emptyset$. Let $f$ be a morphism from $a$ to $b$. Then $f \in \langle a', b' \rangle$ if and only if $\mathcal{Q}[a, b, f]$.

The following proposition is true

(27) Let $A$ be a non empty category structure and $B$ be a non empty substructure of $A$. Then $B$ is full if and only if for all objects $a_1$, $a_2$ of $A$ and for all objects $b_1$, $b_2$ of $B$ such that $b_1 = a_1$ and $b_2 = a_2$ holds $\langle b_1, b_2 \rangle = \langle a_1, a_2 \rangle$.

Now we present two schemes. The scheme *FullSubcategoryEx* deals with a category $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a strict full non empty subcategory $B$ of $\mathcal{A}$ such that for every object $a$ of $\mathcal{A}$ holds $a$ is an object of $B$ if and only if $\mathcal{P}[a]$

provided the parameters satisfy the following condition:

- There exists an object $a$ of $\mathcal{A}$ such that $\mathcal{P}[a]$.

The scheme *FullSubcategoryUniq* deals with a category $\mathcal{A}$, full non empty subcategories $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

The category structure of $\mathcal{B}$ = the category structure of $\mathcal{C}$

provided the parameters meet the following conditions:

- For every object $a$ of $\mathcal{A}$ holds $a$ is an object of $\mathcal{B}$ iff $\mathcal{P}[a]$, and
- For every object $a$ of $\mathcal{A}$ holds $a$ is an object of $\mathcal{C}$ iff $\mathcal{P}[a]$.

## 4. Inclusion Functors and Functor Restrictions

Let $f$ be a function yielding function and let $x$, $y$ be sets. Observe that $f(x, y)$ is relation-like and function-like.

One can prove the following proposition

(28) Let $A$ be a category, $C$ be a non empty subcategory of $A$, and $a$, $b$ be objects of $C$. If $\langle a, b\rangle \neq \emptyset$, then for every morphism $f$ from $a$ to $b$ holds $(\overset{C}{\hookrightarrow})(f) = f$.

Let $A$ be a category and let $C$ be a non empty subcategory of $A$. Note that $\overset{C}{\hookrightarrow}$ is id-preserving and comp-preserving.

Let $A$ be a category and let $C$ be a non empty subcategory of $A$. One can verify that $\overset{C}{\hookrightarrow}$ is precovariant.

Let $A$ be a category and let $C$ be a non empty subcategory of $A$. Then $\overset{C}{\hookrightarrow}$ is a strict covariant functor from $C$ to $A$.

Let $A$, $B$ be categories, let $C$ be a non empty subcategory of $A$, and let $F$ be a covariant functor from $A$ to $B$. Then $F\restriction C$ is a strict covariant functor from $C$ to $B$.

Let $A$, $B$ be categories, let $C$ be a non empty subcategory of $A$, and let $F$ be a contravariant functor from $A$ to $B$. Then $F\restriction C$ is a strict contravariant functor from $C$ to $B$.

Next we state several propositions:

(29) Let $A$, $B$ be categories, $C$ be a non empty subcategory of $A$, $F$ be a functor structure from $A$ to $B$, $a$ be an object of $A$, and $c$ be an object of $C$. If $c = a$, then $(F\restriction C)(c) = F(a)$.

(30) Let $A$, $B$ be categories, $C$ be a non empty subcategory of $A$, $F$ be a covariant functor from $A$ to $B$, $a$, $b$ be objects of $A$, and $c$, $d$ be objects of $C$. Suppose $c = a$ and $d = b$ and $\langle c, d\rangle \neq \emptyset$. Let $f$ be a morphism from $a$ to $b$ and $g$ be a morphism from $c$ to $d$. If $g = f$, then $(F\restriction C)(g) = F(f)$.

(31) Let $A$, $B$ be categories, $C$ be a non empty subcategory of $A$, $F$ be a contravariant functor from $A$ to $B$, $a$, $b$ be objects of $A$, and $c$, $d$ be objects of $C$. Suppose $c = a$ and $d = b$ and $\langle c, d\rangle \neq \emptyset$. Let $f$ be a morphism from $a$ to $b$ and $g$ be a morphism from $c$ to $d$. If $g = f$, then $(F\restriction C)(g) = F(f)$.

(32) Let $A$, $B$ be non empty graphs and $F$ be a bimap structure from $A$ into $B$. Suppose $F$ is precovariant and one-to-one. Let $a$, $b$ be objects of $A$. If $F(a) = F(b)$, then $a = b$.

(33) Let $A$, $B$ be non empty reflexive graphs and $F$ be a feasible precovariant functor structure from $A$ to $B$. Suppose $F$ is faithful. Let $a$, $b$ be objects of $A$. Suppose $\langle a, b\rangle \neq \emptyset$. Let $f$, $g$ be morphisms from $a$ to $b$. If $F(f) = F(g)$, then $f = g$.

(34) Let $A$, $B$ be non empty graphs and $F$ be a precovariant functor structure from $A$ to $B$. Suppose $F$ is surjective. Let $a$, $b$ be objects of $B$. Suppose $\langle a, b\rangle \neq \emptyset$. Let $f$ be a morphism from $a$ to $b$. Then there exist objects $c$, $d$ of $A$ and there exists a morphism $g$ from $c$ to $d$ such that $a = F(c)$ and $b = F(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = F(g)$.

(35) Let $A$, $B$ be non empty graphs and $F$ be a bimap structure from $A$ into $B$. Suppose $F$ is precontravariant and one-to-one. Let $a$, $b$ be objects of $A$. If $F(a) = F(b)$, then $a = b$.

(36) Let $A$, $B$ be non empty reflexive graphs and $F$ be a feasible precontravariant functor structure from $A$ to $B$. Suppose $F$ is faithful. Let $a$, $b$ be objects of $A$. Suppose $\langle a, b\rangle \neq \emptyset$. Let $f$, $g$ be morphisms from $a$ to $b$. If $F(f) = F(g)$, then $f = g$.

(37) Let $A$, $B$ be non empty graphs and $F$ be a precontravariant functor structure from $A$ to $B$. Suppose $F$ is surjective. Let $a$, $b$ be objects of $B$. Suppose $\langle a, b\rangle \neq \emptyset$. Let $f$ be a morphism from $a$ to $b$. Then there exist objects $c$, $d$ of $A$ and there exists a morphism $g$ from $c$ to $d$ such that $b = F(c)$ and $a = F(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = F(g)$.

## 5. Isomorphisms under Arbitrary Functor

Let $A$, $B$ be categories, let $F$ be a functor structure from $A$ to $B$, and let $A'$, $B'$ be categories. We say that $A'$ and $B'$ are isomorphic under $F$ if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) $A'$ is a subcategory of $A$,

(ii) $B'$ is a subcategory of $B$, and

(iii) there exists a covariant functor $G$ from $A'$ to $B'$ such that $G$ is bijective and for every object $a'$ of $A'$ and for every object $a$ of $A$ such that $a' = a$ holds $G(a') = F(a)$ and for all objects $b'$, $c'$ of $A'$ and for all objects $b$, $c$ of $A$ such that $\langle b', c'\rangle \neq \emptyset$ and $b' = b$ and $c' = c$ and for every morphism $f'$ from $b'$ to $c'$ and for every morphism $f$ from $b$ to $c$ such that $f' = f$ holds $G(f') = (\text{Morph-Map}_F(b, c))(f)$.

We say that $A'$ and $B'$ are anti-isomorphic under $F$ if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) $A'$ is a subcategory of $A$,

(ii) $B'$ is a subcategory of $B$, and

(iii) there exists a contravariant functor $G$ from $A'$ to $B'$ such that $G$ is bijective and for every object $a'$ of $A'$ and for every object $a$ of $A$ such that $a' = a$ holds $G(a') = F(a)$ and for all objects $b'$, $c'$ of $A'$ and for all objects $b$, $c$ of $A$ such that $\langle b', c' \rangle \neq \emptyset$ and $b' = b$ and $c' = c$ and for every morphism $f'$ from $b'$ to $c'$ and for every morphism $f$ from $b$ to $c$ such that $f' = f$ holds $G(f') = (\text{Morph-Map}_F(b, c))(f)$.

We now state several propositions:

(38) Let $A$, $B$, $A_1$, $B_1$ be categories and $F$ be a functor structure from $A$ to $B$. If $A_1$ and $B_1$ are isomorphic under $F$, then $A_1$ and $B_1$ are isomorphic.

(39) Let $A$, $B$, $A_1$, $B_1$ be categories and $F$ be a functor structure from $A$ to $B$. Suppose $A_1$ and $B_1$ are anti-isomorphic under $F$. Then $A_1$, $B_1$ are anti-isomorphic.

(40) Let $A$, $B$ be categories and $F$ be a covariant functor from $A$ to $B$. If $A$ and $B$ are isomorphic under $F$, then $F$ is bijective.

(41) Let $A$, $B$ be categories and $F$ be a contravariant functor from $A$ to $B$. If $A$ and $B$ are anti-isomorphic under $F$, then $F$ is bijective.

(42) Let $A$, $B$ be categories and $F$ be a covariant functor from $A$ to $B$. If $F$ is bijective, then $A$ and $B$ are isomorphic under $F$.

(43) Let $A$, $B$ be categories and $F$ be a contravariant functor from $A$ to $B$. If $F$ is bijective, then $A$ and $B$ are anti-isomorphic under $F$.

Now we present two schemes. The scheme *CoBijectRestriction* deals with non empty categories $\mathcal{A}$, $\mathcal{B}$, a covariant functor $\mathcal{C}$ from $\mathcal{A}$ to $\mathcal{B}$, a non empty subcategory $\mathcal{D}$ of $\mathcal{A}$, and a non empty subcategory $\mathcal{E}$ of $\mathcal{B}$, and states that:

$\mathcal{D}$ and $\mathcal{E}$ are isomorphic under $\mathcal{C}$

provided the parameters satisfy the following conditions:

- $\mathcal{C}$ is bijective,
- For every object $a$ of $\mathcal{A}$ holds $a$ is an object of $\mathcal{D}$ iff $\mathcal{C}(a)$ is an object of $\mathcal{E}$, and
- Let $a$, $b$ be objects of $\mathcal{A}$. Suppose $\langle a, b \rangle \neq \emptyset$. Let $a_1$, $b_1$ be objects of $\mathcal{D}$. Suppose $a_1 = a$ and $b_1 = b$. Let $a_2$, $b_2$ be objects of $\mathcal{E}$. Suppose $a_2 = \mathcal{C}(a)$ and $b_2 = \mathcal{C}(b)$. Let $f$ be a morphism from $a$ to $b$. Then $f \in \langle a_1, b_1 \rangle$ if and only if $\mathcal{C}(f) \in \langle a_2, b_2 \rangle$.

The scheme *ContraBijectRestriction* deals with non empty categories $\mathcal{A}$, $\mathcal{B}$, a contravariant functor $\mathcal{C}$ from $\mathcal{A}$ to $\mathcal{B}$, a non empty subcategory $\mathcal{D}$ of $\mathcal{A}$, and a non empty subcategory $\mathcal{E}$ of $\mathcal{B}$, and states that:

$\mathcal{D}$ and $\mathcal{E}$ are anti-isomorphic under $\mathcal{C}$

provided the parameters meet the following conditions:

- $\mathcal{C}$ is bijective,
- For every object $a$ of $\mathcal{A}$ holds $a$ is an object of $\mathcal{D}$ iff $\mathcal{C}(a)$ is an object of $\mathcal{E}$, and

- Let $a$, $b$ be objects of $\mathcal{A}$. Suppose $\langle a, b \rangle \neq \emptyset$. Let $a_1$, $b_1$ be objects of $\mathcal{D}$. Suppose $a_1 = a$ and $b_1 = b$. Let $a_2$, $b_2$ be objects of $\mathcal{E}$. Suppose $a_2 = \mathcal{C}(a)$ and $b_2 = \mathcal{C}(b)$. Let $f$ be a morphism from $a$ to $b$. Then $f \in \langle a_1, b_1 \rangle$ if and only if $\mathcal{C}(f) \in \langle b_2, a_2 \rangle$.

The following propositions are true:

(44) For every category $A$ and for every non empty subcategory $B$ of $A$ holds $B$ and $B$ are isomorphic under $\mathrm{id}_A$.

(45) For all functions $f$, $g$ such that $f \subseteq g$ holds $\curvearrowleft f \subseteq \curvearrowleft g$.

(46) For all functions $f$, $g$ such that $\mathrm{dom}\, f$ is a binary relation and $\curvearrowleft f \subseteq \curvearrowleft g$ holds $f \subseteq g$.

(47) Let $I$, $J$ be sets, $A$ be a many sorted set indexed by $[\!: I, I :\!]$, and $B$ be a many sorted set indexed by $[\!: J, J :\!]$. If $A \mathrel{\dot{\subseteq}} B$, then $\curvearrowleft A \mathrel{\dot{\subseteq}} \curvearrowleft B$.

(48) Let $A$ be a transitive non empty category structure and $B$ be a transitive non empty substructure of $A$. Then $B^{\mathrm{op}}$ is a substructure of $A^{\mathrm{op}}$.

(49) For every category $A$ and for every non empty subcategory $B$ of $A$ holds $B^{\mathrm{op}}$ is a subcategory of $A^{\mathrm{op}}$.

(50) Let $A$ be a category and $B$ be a non empty subcategory of $A$. Then $B$ and $B^{\mathrm{op}}$ are anti-isomorphic under the dualizing functor from $A$ into $A^{\mathrm{op}}$.

(51) Let $A_1$, $A_2$ be categories and $F$ be a covariant functor from $A_1$ to $A_2$. Suppose $F$ is bijective. Let $B_1$ be a non empty subcategory of $A_1$ and $B_2$ be a non empty subcategory of $A_2$. Suppose $B_1$ and $B_2$ are isomorphic under $F$. Then $B_2$ and $B_1$ are isomorphic under $F^{-1}$.

(52) Let $A_1$, $A_2$ be categories and $F$ be a contravariant functor from $A_1$ to $A_2$. Suppose $F$ is bijective. Let $B_1$ be a non empty subcategory of $A_1$ and $B_2$ be a non empty subcategory of $A_2$. Suppose $B_1$ and $B_2$ are anti-isomorphic under $F$. Then $B_2$ and $B_1$ are anti-isomorphic under $F^{-1}$.

(53) Let $A_1$, $A_2$, $A_3$ be categories, $F$ be a covariant functor from $A_1$ to $A_2$, $G$ be a covariant functor from $A_2$ to $A_3$, $B_1$ be a non empty subcategory of $A_1$, $B_2$ be a non empty subcategory of $A_2$, and $B_3$ be a non empty subcategory of $A_3$. Suppose $B_1$ and $B_2$ are isomorphic under $F$ and $B_2$ and $B_3$ are isomorphic under $G$. Then $B_1$ and $B_3$ are isomorphic under $G \cdot F$.

(54) Let $A_1$, $A_2$, $A_3$ be categories, $F$ be a contravariant functor from $A_1$ to $A_2$, $G$ be a covariant functor from $A_2$ to $A_3$, $B_1$ be a non empty subcategory of $A_1$, $B_2$ be a non empty subcategory of $A_2$, and $B_3$ be a non empty subcategory of $A_3$. Suppose $B_1$ and $B_2$ are anti-isomorphic under $F$ and $B_2$ and $B_3$ are isomorphic under $G$. Then $B_1$ and $B_3$ are anti-isomorphic under $G \cdot F$.

(55) Let $A_1$, $A_2$, $A_3$ be categories, $F$ be a covariant functor from $A_1$ to $A_2$, $G$ be a contravariant functor from $A_2$ to $A_3$, $B_1$ be a non empty subcategory

of $A_1$, $B_2$ be a non empty subcategory of $A_2$, and $B_3$ be a non empty subcategory of $A_3$. Suppose $B_1$ and $B_2$ are isomorphic under $F$ and $B_2$ and $B_3$ are anti-isomorphic under $G$. Then $B_1$ and $B_3$ are anti-isomorphic under $G \cdot F$.

(56) Let $A_1$, $A_2$, $A_3$ be categories, $F$ be a contravariant functor from $A_1$ to $A_2$, $G$ be a contravariant functor from $A_2$ to $A_3$, $B_1$ be a non empty subcategory of $A_1$, $B_2$ be a non empty subcategory of $A_2$, and $B_3$ be a non empty subcategory of $A_3$. Suppose $B_1$ and $B_2$ are anti-isomorphic under $F$ and $B_2$ and $B_3$ are anti-isomorphic under $G$. Then $B_1$ and $B_3$ are isomorphic under $G \cdot F$.

## References


[1] Grzegorz Bancerek. Concrete categories. *Formalized Mathematics*, 9(**3**):605–621, 2001.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[7] Artur Korniłowicz. The composition of functors and transformations in alternative categories. *Formalized Mathematics*, 7(**1**):1–7, 1998.
[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[9] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[10] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[11] Andrzej Trybulec. Categories without uniqueness of **cod** and **dom**. *Formalized Mathematics*, 5(**2**):259–267, 1996.
[12] Andrzej Trybulec. Examples of category structures. *Formalized Mathematics*, 5(**4**):493–500, 1996.
[13] Andrzej Trybulec. Functors for alternative categories. *Formalized Mathematics*, 5(**4**):595–608, 1996.
[14] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[15] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[16] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[17] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.


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