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Categorial Background for Duality Theory

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Summary. In the paper, we develop the notation of lattice-wise categories as concrete categories (see [8]) of lattices. Namely, the categories based on [17] with lattices as objects and at least monotone maps between them as morphisms. As examples, we introduce the categories *UPS*, *CONT*, and *ALG* with complete, continuous, and algebraic lattices, respectively, as objects and directed suprema preserving maps as morphisms. Some useful schemes to construct categories of lattices and functors between them are also presented.

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The terminology and notation used in this paper are introduced in the following papers: [17], [18], [12], [20], [9], [14], [4], [19], [1], [15], [21], [22], [16], [10], [11], [6], [7], [13], [2], [3], [8], and [5].

1. Lattice-wise Categories

In this paper x, y are sets.

Let a be a set. a as 1-sorted is a 1-sorted structure and is defined as follows:

(Def. 1) $$a \text{ as 1-sorted} = \begin{cases} a, & \text{if } a \text{ is a 1-sorted structure,} \\ \langle a \rangle, & \text{otherwise.} \end{cases}$$

Let W be a set. The functor POSETS(W) is defined as follows:

(Def. 2) $x \in$ POSETS(W) iff x is a strict poset and the carrier of x as 1-sorted $\in W$.

Let W be a non empty set. One can check that POSETS(W) is non empty.

Let W be a set with non empty elements. Note that POSETS(W) is poset-membered.

Let C be a category. We say that C is carrier-underlaid if and only if:

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(Def. 3) For every object a of C there exists a 1-sorted structure S such that $a = S$ and the carrier of $a =$ the carrier of S.

Let C be a category. We say that C is lattice-wise if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) C is semi-functional and set-id-inheriting,

(ii) every object of C is a lattice, and

(iii) for all objects a, b of C and for all lattices A, B such that $A = a$ and $B = b$ holds $\langle a, b\rangle \subseteq B^{A}_{\leqslant}$.

Let C be a category. We say that C has complete lattices if and only if:

(Def. 5) C is lattice-wise and every object of C is a complete lattice.

One can check that every category which has complete lattices is lattice-wise and every category which is lattice-wise is also concrete and carrier-underlaid.

One can verify that there exists a category which is strict and has complete lattices.

We now state two propositions:

(1) Let C be a carrier-underlaid category and a be an object of C. Then the carrier of $a =$ the carrier of a as 1-sorted.

(2) Let C be a set-id-inheriting carrier-underlaid category and a be an object of C. Then $\mathrm{id}_a = \mathrm{id}_{a \text{ as 1-sorted}}$.

Let C be a lattice-wise category and let a be an object of C. Then a as 1-sorted is a lattice and it can be characterized by the condition:

(Def. 6) a as 1-sorted $= a$.

We introduce $\mathbb{L}_a$ as a synonym of a as 1-sorted.

Let C be a category with complete lattices and let a be an object of C. Then a as 1-sorted is a complete lattice. We introduce $\mathbb{L}_a$ as a synonym of a as 1-sorted.

Let C be a lattice-wise category and let a, b be objects of C. Let us assume that $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. The functor ${}^{@}f$ yielding a monotone map from $\mathbb{L}_a$ into $\mathbb{L}_b$ is defined as follows:

(Def. 7) ${}^{@}f = f$.

The following proposition is true

(3) Let C be a lattice-wise category and a, b, c be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, c\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. Then $g \cdot f = ({}^{@}g) \cdot ({}^{@}f)$.

In this article we present several logical schemes. The scheme *CLCatEx1* deals with a non empty set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a lattice-wise strict category C such that

(i) the carrier of $C = \mathcal{A}$, and

(ii) for all objects a, b of C and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff $\mathcal{P}[\mathbb{L}_a, \mathbb{L}_b, f]$

provided the following conditions are met:

- Every element of $\mathcal{A}$ is a lattice,
- Let a, b, c be lattices. Suppose $a \in \mathcal{A}$ and $b \in \mathcal{A}$ and $c \in \mathcal{A}$. Let f be a map from a into b and g be a map from b into c. If $\mathcal{P}[a, b, f]$ and $\mathcal{P}[b, c, g]$, then $\mathcal{P}[a, c, g \cdot f]$, and
- For every lattice a such that $a \in \mathcal{A}$ holds $\mathcal{P}[a, a, \mathrm{id}_a]$.

The scheme *CLCatEx2* deals with a non empty set $\mathcal{A}$, a unary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

There exists a lattice-wise strict category C such that

(i) for every lattice x holds x is an object of C iff x is strict and $\mathcal{P}[x]$ and the carrier of $x \in \mathcal{A}$, and

(ii) for all objects a, b of C and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff $\mathcal{Q}[\mathbb{L}_a, \mathbb{L}_b, f]$

provided the parameters satisfy the following conditions:

- There exists a strict lattice x such that $\mathcal{P}[x]$ and the carrier of $x \in \mathcal{A}$,
- Let a, b, c be lattices. Suppose $\mathcal{P}[a]$ and $\mathcal{P}[b]$ and $\mathcal{P}[c]$. Let f be a map from a into b and g be a map from b into c. If $\mathcal{Q}[a, b, f]$ and $\mathcal{Q}[b, c, g]$, then $\mathcal{Q}[a, c, g \cdot f]$, and
- For every lattice a such that $\mathcal{P}[a]$ holds $\mathcal{Q}[a, a, \mathrm{id}_a]$.

The scheme *CLCatUniq1* deals with a non empty set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

Let C_1, C_2 be lattice-wise categories. Suppose that

(i) the carrier of $C_1 = \mathcal{A}$,

(ii) for all objects a, b of C_1 and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff $\mathcal{P}[a, b, f]$,

(iii) the carrier of $C_2 = \mathcal{A}$, and

(iv) for all objects a, b of C_2 and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff $\mathcal{P}[a, b, f]$.

Then the category structure of C_1 = the category structure of C_2

for all values of the parameters.

The scheme *CLCatUniq2* deals with a non empty set $\mathcal{A}$, a unary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

Let C_1, C_2 be lattice-wise categories. Suppose that

(i) for every lattice x holds x is an object of C_1 iff x is strict and $\mathcal{P}[x]$ and the carrier of $x \in \mathcal{A}$,

(ii) for all objects a, b of C_1 and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff $\mathcal{Q}[a, b, f]$,

(iii) for every lattice x holds x is an object of C_2 iff x is strict and $\mathcal{P}[x]$ and the carrier of $x \in \mathcal{A}$, and

(iv) for all objects a, b of C_2 and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b\rangle$ iff $\mathcal{Q}[a, b, f]$.

Then the category structure of $C_1 =$ the category structure of C_2

for all values of the parameters.

The scheme *CLCovariantFunctorEx* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a lattice, a ternary functor $\mathcal{G}$ yielding a function, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

There exists a covariant strict functor F from $\mathcal{A}$ to $\mathcal{B}$ such that

(i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(\mathbb{L}_a)$, and

(ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(\mathbb{L}_a, \mathbb{L}_b, {}^{@}f)$

provided the parameters meet the following conditions:

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{A}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[a, b, f]$,
- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{B}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{B}$ and $b \in$ the carrier of $\mathcal{B}$ and $\mathcal{Q}[a, b, f]$,
- For every lattice a such that $a \in$ the carrier of $\mathcal{A}$ holds $\mathcal{F}(a) \in$ the carrier of $\mathcal{B}$,
- Let a, b be lattices and f be a map from a into b. If $\mathcal{P}[a, b, f]$, then $\mathcal{G}(a, b, f)$ is a map from $\mathcal{F}(a)$ into $\mathcal{F}(b)$ and $\mathcal{Q}[\mathcal{F}(a), \mathcal{F}(b), \mathcal{G}(a, b, f)]$,
- For every lattice a such that $a \in$ the carrier of $\mathcal{A}$ holds $\mathcal{G}(a, a, \mathrm{id}_a) = \mathrm{id}_{\mathcal{F}(a)}$, and
- Let a, b, c be lattices, f be a map from a into b, and g be a map from b into c. If $\mathcal{P}[a, b, f]$ and $\mathcal{P}[b, c, g]$, then $\mathcal{G}(a, c, g \cdot f) = \mathcal{G}(b, c, g) \cdot \mathcal{G}(a, b, f)$.

The scheme *CLContravariantFunctorEx* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a lattice, a ternary functor $\mathcal{G}$ yielding a function, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

There exists a contravariant strict functor F from $\mathcal{A}$ to $\mathcal{B}$ such that

(i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(\mathbb{L}_a)$, and

(ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(\mathbb{L}_a, \mathbb{L}_b, {}^{@}f)$

provided the parameters satisfy the following conditions:

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{A}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[a, b, f]$,

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{B}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{B}$ and $b \in$ the carrier of $\mathcal{B}$ and $\mathcal{Q}[a, b, f]$,
- For every lattice a such that $a \in$ the carrier of $\mathcal{A}$ holds $\mathcal{F}(a) \in$ the carrier of $\mathcal{B}$,
- Let a, b be lattices and f be a map from a into b. If $\mathcal{P}[a, b, f]$, then $\mathcal{G}(a, b, f)$ is a map from $\mathcal{F}(b)$ into $\mathcal{F}(a)$ and $\mathcal{Q}[\mathcal{F}(b), \mathcal{F}(a), \mathcal{G}(a, b, f)]$,
- For every lattice a such that $a \in$ the carrier of $\mathcal{A}$ holds $\mathcal{G}(a, a, \mathrm{id}_a) = \mathrm{id}_{\mathcal{F}(a)}$, and
- Let a, b, c be lattices, f be a map from a into b, and g be a map from b into c. If $\mathcal{P}[a, b, f]$ and $\mathcal{P}[b, c, g]$, then $\mathcal{G}(a, c, g \cdot f) = \mathcal{G}(a, b, f) \cdot \mathcal{G}(b, c, g)$.

The scheme *CLCatIsomorphism* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a lattice, a ternary functor $\mathcal{G}$ yielding a function, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{A}$ and $\mathcal{B}$ are isomorphic

provided the parameters meet the following conditions:

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{A}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[a, b, f]$,
- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{B}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{B}$ and $b \in$ the carrier of $\mathcal{B}$ and $\mathcal{Q}[a, b, f]$,
- There exists a covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a, b, f)$,
- For all lattices a, b such that $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ holds if $\mathcal{F}(a) = \mathcal{F}(b)$, then $a = b$,
- For all lattices a, b and for all maps f, g from a into b such that $\mathcal{P}[a, b, f]$ and $\mathcal{P}[a, b, g]$ holds if $\mathcal{G}(a, b, f) = \mathcal{G}(a, b, g)$, then $f = g$, and
- Let a, b be lattices and f be a map from a into b. Suppose $\mathcal{Q}[a, b, f]$. Then there exist lattices c, d and there exists a map g from c into d such that $c \in$ the carrier of $\mathcal{A}$ and $d \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[c, d, g]$ and $a = \mathcal{F}(c)$ and $b = \mathcal{F}(d)$ and $f = \mathcal{G}(c, d, g)$.

The scheme *CLCatAntiIsomorphism* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a lattice, a ternary functor $\mathcal{G}$ yielding a function, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{A}$, $\mathcal{B}$ are anti-isomorphic

provided the following conditions are met:

- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{A}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[a, b, f]$,
- Let a, b be lattices and f be a map from a into b. Then $f \in$ (the arrows of $\mathcal{B}$)(a, b) if and only if $a \in$ the carrier of $\mathcal{B}$ and $b \in$ the carrier of $\mathcal{B}$ and $\mathcal{Q}[a, b, f]$,
- There exists a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a, b, f)$,
- For all lattices a, b such that $a \in$ the carrier of $\mathcal{A}$ and $b \in$ the carrier of $\mathcal{A}$ holds if $\mathcal{F}(a) = \mathcal{F}(b)$, then $a = b$,
- For all lattices a, b and for all maps f, g from a into b such that $\mathcal{G}(a, b, f) = \mathcal{G}(a, b, g)$ holds $f = g$, and
- Let a, b be lattices and f be a map from a into b. Suppose $\mathcal{Q}[a, b, f]$. Then there exist lattices c, d and there exists a map g from c into d such that $c \in$ the carrier of $\mathcal{A}$ and $d \in$ the carrier of $\mathcal{A}$ and $\mathcal{P}[c, d, g]$ and $b = \mathcal{F}(c)$ and $a = \mathcal{F}(d)$ and $f = \mathcal{G}(c, d, g)$.

2. Equivalence of Lattice-wise Categories

Let C be a lattice-wise category. We say that C has all isomorphisms if and only if:

(Def. 8) For all objects a, b of C and for every map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ such that f is isomorphic holds $f \in \langle a, b\rangle$.

One can verify that there exists a strict lattice-wise category which has all isomorphisms.

The following propositions are true:

(4) Let C be a lattice-wise category with all isomorphisms, a, b be objects of C, and f be a morphism from a to b. If ${}^{@}f$ is isomorphic, then f is iso.

(5) Let C be a lattice-wise category and a, b be objects of C. Suppose $\langle a, b\rangle \neq \emptyset$ and $\langle b, a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is iso, then ${}^{@}f$ is isomorphic.

The scheme *CLCatEquivalence* deals with lattice-wise categories $\mathcal{A}$, $\mathcal{B}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding lattices, two ternary functors $\mathcal{H}$ and $\mathcal{I}$ yielding functions, two unary functors $\mathcal{A}$ and $\mathcal{B}$ yielding functions, and two ternary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{A}$ and $\mathcal{B}$ are equivalent

provided the parameters satisfy the following conditions:

- For all objects a, b of $\mathcal{A}$ and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff $\mathcal{P}[\mathbb{L}_a, \mathbb{L}_b, f]$,
- For all objects a, b of $\mathcal{B}$ and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff $\mathcal{Q}[\mathbb{L}_a, \mathbb{L}_b, f]$,
- There exists a covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{H}(a, b, f)$,
- There exists a covariant functor G from $\mathcal{B}$ to $\mathcal{A}$ such that
  (i) for every object a of $\mathcal{B}$ holds $G(a) = \mathcal{G}(a)$, and
  (ii) for all objects a, b of $\mathcal{B}$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $G(f) = \mathcal{I}(a, b, f)$,
- Let a be a lattice. Suppose $a \in$ the carrier of $\mathcal{A}$. Then there exists a monotone map f from $\mathcal{G}(\mathcal{F}(a))$ into a such that $f = \mathcal{A}(a)$ and f is isomorphic and $\mathcal{P}[\mathcal{G}(\mathcal{F}(a)), a, f]$ and $\mathcal{P}[a, \mathcal{G}(\mathcal{F}(a)), f^{-1}]$,
- Let a be a lattice. Suppose $a \in$ the carrier of $\mathcal{B}$. Then there exists a monotone map f from a into $\mathcal{F}(\mathcal{G}(a))$ such that $f = \mathcal{B}(a)$ and f is isomorphic and $\mathcal{Q}[a, \mathcal{F}(\mathcal{G}(a)), f]$ and $\mathcal{Q}[\mathcal{F}(\mathcal{G}(a)), a, f^{-1}]$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{A}(b) \cdot \mathcal{I}(\mathcal{F}(a), \mathcal{F}(b), \mathcal{H}(a, b, f)) = ({}^{@}f) \cdot \mathcal{A}(a)$, and
- For all objects a, b of $\mathcal{B}$ such that $\langle a, b \rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{H}(\mathcal{G}(a), \mathcal{G}(b), \mathcal{I}(a, b, f)) \cdot \mathcal{B}(a) = \mathcal{B}(b) \cdot ({}^{@}f)$.

3. *UPS* Category

Let R be a binary relation. We say that R is upper-bounded if and only if:

(Def. 9) There exists x such that for every y such that $y \in \operatorname{field} R$ holds $\langle y, x \rangle \in R$.

Let us note that every binary relation which is well-ordering is also reflexive, transitive, antisymmetric, connected, and well founded.

Let us mention that there exists a binary relation which is well-ordering.

Next we state the proposition

(6) Let f be an one-to-one function and R be a binary relation. Then $\langle x, y \rangle \in f \cdot R \cdot f^{-1}$ if and only if $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} f$ and $\langle f(x), f(y) \rangle \in R$.

Let f be an one-to-one function and let R be a reflexive binary relation. Note that $f \cdot R \cdot f^{-1}$ is reflexive.

Let f be an one-to-one function and let R be an antisymmetric binary relation. Note that $f \cdot R \cdot f^{-1}$ is antisymmetric.

Let f be an one-to-one function and let R be a transitive binary relation. Note that $f \cdot R \cdot f^{-1}$ is transitive.

Next we state the proposition

(7) Let X be a set and A be an ordinal number. If $X \approx A$, then there exists an order R in X such that R well orders X and $\overline{R} = A$.

Let X be a non empty set. Observe that there exists an order in X which is upper-bounded and well-ordering.

Next we state four propositions:

(8) Let P be a reflexive non empty relational structure. Then P is upper-bounded if and only if the internal relation of P is upper-bounded.

(9) Let P be an upper-bounded non empty poset. Suppose the internal relation of P is well-ordering. Then P is connected, complete, and continuous.

(10) Let P be an upper-bounded non empty poset. Suppose the internal relation of P is well-ordering. Let x, y be elements of P. If $y < x$, then there exists an element z of P such that z is compact and $y \leqslant z$ and $z \leqslant x$.

(11) Let P be an upper-bounded non empty poset. If the internal relation of P is well-ordering, then P is algebraic.

Let X be a non empty set and let R be an upper-bounded well-ordering order in X. Observe that $\langle X, R \rangle$ is complete connected continuous and algebraic.

Let us observe that every set which is non trivial has a non-empty element.

Let W be a non empty set. Let us assume that there exists an element w of W such that w is non empty. The functor UPS_W yielding a lattice-wise strict category is defined by the conditions (Def. 10).

(Def. 10)(i) For every lattice x holds x is an object of UPS_W iff x is strict and complete and the carrier of $x \in W$, and

(ii) for all objects a, b of UPS_W and for every monotone map f from $\mathbb{L}_a$ into $\mathbb{L}_b$ holds $f \in \langle a, b \rangle$ iff f is directed-sups-preserving.

Let W be a set with a non-empty element. Observe that UPS_W has complete lattices and all isomorphisms.

One can prove the following four propositions:

(12) For every set W with a non-empty element holds the carrier of $UPS_W \subseteq$ POSETS(W).

(13) Let W be a set with a non-empty element and given x. Then x is an object of UPS_W if and only if x is a complete lattice and $x \in$ POSETS(W).

(14) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of UPS_W if and only if L is strict and complete.

(15) Let W be a set with a non-empty element, a, b be objects of UPS_W, and f be a set. Then $f \in \langle a, b\rangle$ if and only if f is a directed-sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

Let W be a set with a non-empty element and let a, b be objects of UPS_W. Observe that $\langle a, b\rangle$ is non empty.

4. Lattice-wise Subcategories

Next we state the proposition

(16) Let A be a category, B be a non empty subcategory of A, a be an object of A, and b be an object of B. If $b = a$, then the carrier of $b =$ the carrier of a.

Let A be a set-id-inheriting category. Observe that every non empty subcategory of A is set-id-inheriting.

Let A be a para-functional category. One can verify that every non empty subcategory of A is para-functional.

Let A be a semi-functional category. Note that every non empty transitive substructure of A is semi-functional.

Let A be a carrier-underlaid category. Note that every non empty subcategory of A is carrier-underlaid.

Let A be a lattice-wise category. Observe that every non empty subcategory of A is lattice-wise.

Let A be a lattice-wise category with all isomorphisms. Observe that every non empty subcategory of A which is full has all isomorphisms.

Let A be a category with complete lattices. One can check that every non empty subcategory of A has complete lattices.

Let W be a set with a non-empty element. The functor $CONT_W$ yielding a strict full non empty subcategory of UPS_W is defined by:

(Def. 11) For every object a of UPS_W holds a is an object of $CONT_W$ iff $\mathbb{L}_a$ is continuous.

Let W be a set with a non-empty element. The functor ALG_W yielding a strict full non empty subcategory of $CONT_W$ is defined by:

(Def. 12) For every object a of $CONT_W$ holds a is an object of ALG_W iff $\mathbb{L}_a$ is algebraic.

The following four propositions are true:

(17) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of $CONT_W$ if and only if L is strict, complete, and continuous.

(18) Let W be a set with a non-empty element and L be a lattice. Suppose the carrier of $L \in W$. Then L is an object of ALG_W if and only if L is strict, complete, and algebraic.

(19) Let W be a set with a non-empty element, a, b be objects of $CONT_W$, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is a directed-sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

(20) Let W be a set with a non-empty element, a, b be objects of ALG_W, and f be a set. Then $f \in \langle a, b \rangle$ if and only if f is a directed-sups-preserving map from $\mathbb{L}_a$ into $\mathbb{L}_b$.

Let W be a set with a non-empty element and let a, b be objects of $CONT_W$. One can check that $\langle a, b \rangle$ is non empty.

Let W be a set with a non-empty element and let a, b be objects of ALG_W. One can check that $\langle a, b \rangle$ is non empty.

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