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Fan Homeomorphisms in the Plane

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Summary. We will introduce four homeomorphisms (Fan morphisms) which give spoke-like distortion to the plane. They do not change the norms of vectors and preserve halfplanes invariant. These morphisms are used to regulate placement of points on the circle.

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The articles [14], [18], [5], [7], [1], [2], [11], [12], [10], [3], [13], [4], [9], [19], [16], [17], [15], [8], and [6] provide the notation and terminology for this paper.

1. Preliminaries

In this paper x, a denote real numbers and p, q denote points of $\mathcal{E}^2_{\rm T}$.

The following propositions are true:

(1) If $|x| < a$, then $-a < x$ and $x < a$.

(2) If $a \geqslant 0$ and $(x - a) \cdot (x + a) < 0$, then $-a < x$ and $x < a$.

(3) For every real number s_1 such that $-1 < s_1$ and $s_1 < 1$ holds $1 + s_1 > 0$ and $1 - s_1 > 0$.

(4) For every real number a such that $a^{\mathbf{2}} \leqslant 1$ holds $-1 \leqslant a$ and $a \leqslant 1$.

(5) For every real number a such that $a^{\mathbf{2}} < 1$ holds $-1 < a$ and $a < 1$.

(6) Let X be a non empty topological structure, g be a map from X into $\mathbb{R}^{\mathbf{1}}$, B be a subset of X, and a be a real number. If g is continuous and $B = \{p; p$ ranges over points of X: $\pi_p g > a\}$, then B is open.

(7) Let X be a non empty topological structure, g be a map from X into $\mathbb{R}^{\mathbf{1}}$, B be a subset of X, and a be a real number. If g is continuous and $B = \{p; p$ ranges over points of X: $\pi_p g < a\}$, then B is open.

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(8) Let f be a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$. Suppose that

(i) f is continuous and one-to-one,

(ii) $\operatorname{rng} f = \Omega_{\mathcal{E}^2_{\rm T}}$, and

(iii) for every point p_2 of $\mathcal{E}^2_{\rm T}$ there exists a non empty compact subset K of $\mathcal{E}^2_{\rm T}$ such that $K = f^\circ K$ and there exists a subset V_2 of $\mathcal{E}^2_{\rm T}$ such that $p_2 \in V_2$ and V_2 is open and $V_2 \subseteq K$ and $f(p_2) \in V_2$.

Then f is a homeomorphism.

(9) Let X be a non empty topological space, f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$, and a, b be real numbers. Suppose f_1 is continuous and f_2 is continuous and $b \neq 0$ and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = \frac{\frac{r_1}{r_2} - a}{b}$, and

(ii) g is continuous.

(10) Let X be a non empty topological space, f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$, and a, b be real numbers. Suppose f_1 is continuous and f_2 is continuous and $b \neq 0$ and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_2 \cdot \frac{\frac{r_1}{r_2} - a}{b}$, and

(ii) g is continuous.

(11) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = {r_1}^{\mathbf{2}}$ and g is continuous.

(12) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = |r_1|$ and g is continuous.

(13) Let X be a non empty topological space and f_1 be a map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose f_1 is continuous. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of X and for every real number r_1 such that $f_1(p) = r_1$ holds $g(p) = -r_1$ and g is continuous.

(14) Let X be a non empty topological space, f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$, and a, b be real numbers. Suppose f_1 is continuous and f_2 is continuous and $b \neq 0$ and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_2 \cdot -\sqrt{|1 - (\frac{\frac{r_1}{r_2} - a}{b})^{\mathbf{2}}|}$, and

(ii) g is continuous.

(15) Let X be a non empty topological space, f_1, f_2 be maps from X into $\mathbb{R}^{\mathbf{1}}$, and a, b be real numbers. Suppose f_1 is continuous and f_2 is continuous and $b \neq 0$ and for every point q of X holds $f_2(q) \neq 0$. Then there exists a map g from X into $\mathbb{R}^{\mathbf{1}}$ such that

(i) for every point p of X and for all real numbers r_1, r_2 such that $f_1(p) = r_1$ and $f_2(p) = r_2$ holds $g(p) = r_2 \cdot \sqrt{|1 - (\frac{\frac{r_1}{r_2} - a}{b})^{\mathbf{2}}|}$, and

(ii) g is continuous.

Let n be a natural number. The functor $n\,\mathrm{NormF}$ yields a function from the carrier of $\mathcal{E}^n_{\mathrm{T}}$ into the carrier of $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def. 1) For every point q of $\mathcal{E}^n_{\mathrm{T}}$ holds $n\,\mathrm{NormF}(q) = |q|$.

Next we state several propositions:

(16) For every natural number n holds $\mathrm{dom}(n\,\mathrm{NormF}) =$ the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and $\mathrm{dom}(n\,\mathrm{NormF}) = \mathcal{R}^n$.

(18)[1] For every natural number n and for all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ holds $||p| - |q|| \leqslant |p - q|$.

(19) For every natural number n and for every map f from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$ such that $f = n\,\mathrm{NormF}$ holds f is continuous.

(20) Let n be a natural number, K_0 be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^n_{\mathrm{T}}) \restriction K_0$ into $\mathbb{R}^{\mathbf{1}}$. If for every point p of $(\mathcal{E}^n_{\mathrm{T}}) \restriction K_0$ holds $f(p) = n\,\mathrm{NormF}(p)$, then f is continuous.

(21) Let n be a natural number, p be a point of $\mathcal{E}^n$, r be a real number, and B be a subset of $\mathcal{E}^n_{\mathrm{T}}$. If $B = \overline{\mathrm{Ball}}(p, r)$, then B is Bounded and closed.

(22) For every point p of $\mathcal{E}^2$ and for every real number r and for every subset B of $\mathcal{E}^2_{\mathrm{T}}$ such that $B = \overline{\mathrm{Ball}}(p, r)$ holds B is compact.

2. Fan Morphism for West

Let s be a real number and let q be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{FanW}(s, q)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 2) $$\mathrm{FanW}(s, q) = \begin{cases} |q| \cdot [-\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1 - s})^{\mathbf{2}}}, \frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1 - s}], & \text{if } \frac{q_{\mathbf{2}}}{|q|} \geqslant s \text{ and } q_{\mathbf{1}} < 0, \\ |q| \cdot [-\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1 + s})^{\mathbf{2}}}, \frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1 + s}], & \text{if } \frac{q_{\mathbf{2}}}{|q|} < s \text{ and } q_{\mathbf{1}} < 0, \\ q, & \text{otherwise.} \end{cases}$$

Let s be a real number. The functor s-FanMorphW yields a function from the carrier of $\mathcal{E}^2_{\mathrm{T}}$ into the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

[1] The proposition (17) has been removed.

(Def. 3) For every point q of $\mathcal{E}^2_{\rm T}$ holds s-FanMorphW$(q) = \mathrm{FanW}(s, q)$.

Next we state a number of propositions:

(23) Let s_1 be a real number. Then

(i) if $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q_{\mathbf{1}} < 0$, then s_1-FanMorphW$(q) = [|q| \cdot -\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1 - s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1 - s_1}]$, and

(ii) if $q_{\mathbf{1}} \geqslant 0$, then s_1-FanMorphW$(q) = q$.

(24) For every real number s_1 such that $\frac{q_{\mathbf{2}}}{|q|} \leqslant s_1$ and $q_{\mathbf{1}} < 0$ holds s_1-FanMorphW$(q) = [|q| \cdot -\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1 + s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1 + s_1}]$.

(25) Let s_1 be a real number such that $-1 < s_1$ and $s_1 < 1$. Then

(i) if $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q_{\mathbf{1}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$, then s_1-FanMorphW$(q) = [|q| \cdot -\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1 - s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1 - s_1}]$, and

(ii) if $\frac{q_{\mathbf{2}}}{|q|} \leqslant s_1$ and $q_{\mathbf{1}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$, then s_1-FanMorphW$(q) = [|q| \cdot -\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1 + s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1 + s_1}]$.

(26) Let s_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1 - s_1}$, and

(iv) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\mathbf{1}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then f is continuous.

(27) Let s_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1 + s_1}$, and

(iv) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\mathbf{1}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then f is continuous.

(28) Let s_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot -\sqrt{1 - (\frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1 - s_1})^{\mathbf{2}}}$, and

(iv) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \leqslant 0$ and $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then f is continuous.

(29) Let s_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot -\sqrt{1 - (\frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1 + s_1})^{\mathbf{2}}}$, and

(iv) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \leqslant 0$ and $\frac{q_{\mathbf{2}}}{|q|} \leqslant s_1$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then f is continuous.

(30) Let s_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} \leqslant 0 \ \wedge \ q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{2}}}{|p|} \geqslant s_1 \ \wedge \ p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous.

(31) Let s_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $q_{\mathbf{1}} \leqslant 0 \ \wedge \ q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{2}}}{|p|} \leqslant s_1 \ \wedge \ p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous.

(32) For every real number s_1 and for every subset K_3 of $\mathcal{E}^2_{\mathrm{T}}$ such that $K_3 = \{p : p_{\mathbf{2}} \geqslant s_1 \cdot |p| \ \wedge \ p_{\mathbf{1}} \leqslant 0\}$ holds K_3 is closed.

(33) For every real number s_1 and for every subset K_3 of $\mathcal{E}^2_{\mathrm{T}}$ such that $K_3 = \{p : p_{\mathbf{2}} \leqslant s_1 \cdot |p| \ \wedge \ p_{\mathbf{1}} \leqslant 0\}$ holds K_3 is closed.

(34) Let s_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous.

(35) Let s_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_0$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then f is continuous.

(36) Let B_0 be a subset of $\mathcal{E}^2_{\mathrm{T}}$ and K_0 be a subset of $(\mathcal{E}^2_{\mathrm{T}}) \restriction B_0$. Suppose $B_0 =$ (the carrier of $\mathcal{E}^2_{\mathrm{T}}) \setminus \{0_{\mathcal{E}^2_{\mathrm{T}}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}\}$. Then K_0 is

closed.

(37) Let s_1 be a real number, B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(38) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$ and K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then K_0 is closed.

(39) Let s_1 be a real number, B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphW $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(40) For every real number s_1 and for every point p of $\mathcal{E}^2_{\rm T}$ holds $|s_1$-FanMorphW$(p)| = |p|$.

(41) For every real number s_1 and for all sets x, K_0 such that $-1 < s_1$ and $s_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds s_1-FanMorphW$(x) \in K_0$.

(42) For every real number s_1 and for all sets x, K_0 such that $-1 < s_1$ and $s_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds s_1-FanMorphW$(x) \in K_0$.

(43) Let s_1 be a real number and D be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $D^{\rm c} = \{0_{\mathcal{E}^2_{\rm T}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\rm T})\restriction D$ into $(\mathcal{E}^2_{\rm T})\restriction D$ such that $h = s_1$-FanMorphW $\restriction D$ and h is continuous.

(44) Let s_1 be a real number. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a map h from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $h = s_1$-FanMorphW and h is continuous.

(45) For every real number s_1 such that $-1 < s_1$ and $s_1 < 1$ holds s_1-FanMorphW is one-to-one.

(46) For every real number s_1 such that $-1 < s_1$ and $s_1 < 1$ holds s_1-FanMorphW is a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and rng(s_1-FanMorphW) = the carrier of $\mathcal{E}^2_{\rm T}$.

(47) Let s_1 be a real number and p_2 be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a non empty compact subset K of $\mathcal{E}^2_{\rm T}$ such that $K = s_1$-FanMorphW$^\circ K$ and there exists a subset V_2 of $\mathcal{E}^2_{\rm T}$ such that $p_2 \in V_2$ and V_2 is open and $V_2 \subseteq K$ and s_1-FanMorphW$(p_2) \in V_2$.

(48) Let s_1 be a real number. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $f = s_1$-FanMorphW and f is a homeomorphism.

(49) Let s_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$

and $s_1 < 1$ and $q_{\mathbf{1}} < 0$ and $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_1$-FanMorphW(q), then $p_{\mathbf{1}} < 0$ and $p_{\mathbf{2}} \geqslant 0$.

(50) Let s_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\mathbf{1}} < 0$ and $\frac{q_{\mathbf{2}}}{|q|} < s_1$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_1$-FanMorphW(q), then $p_{\mathbf{1}} < 0$ and $p_{\mathbf{2}} < 0$.

(51) Let s_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} < 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} \geqslant s_1$ and $(q_2)_{\mathbf{1}} < 0$ and $\frac{(q_2)_{\mathbf{2}}}{|q_2|} \geqslant s_1$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_1$-FanMorphW(q_1) and $p_2 = s_1$-FanMorphW(q_2), then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(52) Let s_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} < 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < s_1$ and $(q_2)_{\mathbf{1}} < 0$ and $\frac{(q_2)_{\mathbf{2}}}{|q_2|} < s_1$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_1$-FanMorphW(q_1) and $p_2 = s_1$-FanMorphW(q_2), then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(53) Let s_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} < 0$ and $(q_2)_{\mathbf{1}} < 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = s_1$-FanMorphW(q_1) and $p_2 = s_1$-FanMorphW(q_2), then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(54) Let s_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\mathbf{1}} < 0$ and $\frac{q_{\mathbf{2}}}{|q|} = s_1$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = s_1$-FanMorphW(q), then $p_{\mathbf{1}} < 0$ and $p_{\mathbf{2}} = 0$.

(55) For every real number s_1 holds $0_{\mathcal{E}^2_{\mathrm{T}}} = s_1$-FanMorphW$(0_{\mathcal{E}^2_{\mathrm{T}}})$.

3. Fan Morphism for North

Let s be a real number and let q be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor FanN$(s,q)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 4) $$\mathrm{FanN}(s,q) = \begin{cases} |q| \cdot [\frac{\frac{q_{\mathbf{1}}}{|q|} - s}{1-s}, \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - s}{1-s})^{\mathbf{2}}}], & \text{if } \frac{q_{\mathbf{1}}}{|q|} \geqslant s \text{ and } q_{\mathbf{2}} > 0, \\ |q| \cdot [\frac{\frac{q_{\mathbf{1}}}{|q|} - s}{1+s}, \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - s}{1+s})^{\mathbf{2}}}], & \text{if } \frac{q_{\mathbf{1}}}{|q|} < s \text{ and } q_{\mathbf{2}} > 0, \\ q, & \text{otherwise.} \end{cases}$$

Let c be a real number. The functor c-FanMorphN yielding a function from the carrier of $\mathcal{E}^2_{\mathrm{T}}$ into the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 5) For every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds c-FanMorphN$(q) = $ FanN(c,q).

One can prove the following propositions:

(56) Let c_1 be a real number. Then

(i) if $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q_{\mathbf{2}} > 0$, then c_1-FanMorphN$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 - c_1}, |q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 - c_1})^{\mathbf{2}}}]$, and

(ii) if $q_{\mathbf{2}} \leqslant 0$, then c_1-FanMorphN$(q) = q$.

(57) For every real number c_1 such that $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q_{\mathbf{2}} > 0$ holds c_1-FanMorphN$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 + c_1}, |q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 + c_1})^{\mathbf{2}}}]$.

(58) Let c_1 be a real number such that $-1 < c_1$ and $c_1 < 1$. Then

(i) if $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q_{\mathbf{2}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then c_1-FanMorphN$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 - c_1}$, $|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 - c_1})^{\mathbf{2}}}]$, and

(ii) if $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q_{\mathbf{2}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then c_1-FanMorphN$(q) = [|q| \cdot \frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 + c_1}$, $|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{1}}}{|q|} - c_1}{1 + c_1})^{\mathbf{2}}}]$.

(59) Let c_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 - c_1}$, and

(iv) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{2}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then f is continuous.

(60) Let c_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 + c_1}$, and

(iv) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{2}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then f is continuous.

(61) Let c_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < c_1$,

(ii) $c_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot \sqrt{1 - (\frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 - c_1})^{\mathbf{2}}}$, and

(iv) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\mathbf{2}} \geqslant 0$ and $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.
Then f is continuous.

(62) Let c_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
(i) $-1 < c_1$,
(ii) $c_1 < 1$,
(iii) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $f(p) = |p| \cdot \sqrt{1 - (\frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 + c_1})^{\mathbf{2}}}$, and
(iv) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T}) \restriction K_1$ holds $q_{\mathbf{2}} \geqslant 0$ and $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.
Then f is continuous.

(63) Let c_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\mathbf{2}} \geqslant 0 \wedge q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{1}}}{|p|} \geqslant c_1 \wedge p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(64) Let c_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\mathbf{2}} \geqslant 0 \wedge q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{1}}}{|p|} \leqslant c_1 \wedge p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(65) For every real number c_1 and for every subset K_3 of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\mathbf{1}} \geqslant c_1 \cdot |p| \wedge p_{\mathbf{2}} \geqslant 0\}$ holds K_3 is closed.

(66) For every real number c_1 and for every subset K_3 of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\mathbf{1}} \leqslant c_1 \cdot |p| \wedge p_{\mathbf{2}} \geqslant 0\}$ holds K_3 is closed.

(67) Let c_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(68) Let c_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T}) \restriction K_0$ into $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(69) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$ and K_0 be a subset of $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then K_0 is closed.

(70) Let B_0 be a subset of $\mathcal{E}^2_{\rm T}$ and K_0 be a subset of $(\mathcal{E}^2_{\rm T}) \restriction B_0$. Suppose $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then K_0 is

closed.

(71) Let c_1 be a real number, B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(72) Let c_1 be a real number, B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphN $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(73) For every real number c_1 and for every point p of $\mathcal{E}^2_{\rm T}$ holds $|c_1$-FanMorphN$(p)| = |p|$.

(74) For every real number c_1 and for all sets x, K_0 such that $-1 < c_1$ and $c_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{2}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds c_1-FanMorphN$(x) \in K_0$.

(75) For every real number c_1 and for all sets x, K_0 such that $-1 < c_1$ and $c_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{2}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds c_1-FanMorphN$(x) \in K_0$.

(76) Let c_1 be a real number and D be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $D^{\rm c} = \{0_{\mathcal{E}^2_{\rm T}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\rm T})\restriction D$ into $(\mathcal{E}^2_{\rm T})\restriction D$ such that $h = c_1$-FanMorphN $\restriction D$ and h is continuous.

(77) Let c_1 be a real number. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a map h from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $h = c_1$-FanMorphN and h is continuous.

(78) For every real number c_1 such that $-1 < c_1$ and $c_1 < 1$ holds c_1-FanMorphN is one-to-one.

(79) For every real number c_1 such that $-1 < c_1$ and $c_1 < 1$ holds c_1-FanMorphN is a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and rng(c_1-FanMorphN) = the carrier of $\mathcal{E}^2_{\rm T}$.

(80) Let c_1 be a real number and p_2 be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a non empty compact subset K of $\mathcal{E}^2_{\rm T}$ such that $K = c_1$-FanMorphN$^\circ$ K and there exists a subset V_2 of $\mathcal{E}^2_{\rm T}$ such that $p_2 \in V_2$ and V_2 is open and $V_2 \subseteq K$ and c_1-FanMorphN$(p_2) \in V_2$.

(81) Let c_1 be a real number. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $f = c_1$-FanMorphN and f is a homeomorphism.

(82) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} > 0$ and $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} > 0$ and $p_{\mathbf{1}} \geqslant 0$.

(83) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 <$

1 and $q_{\mathbf{2}} > 0$ and $\frac{q_{\mathbf{1}}}{|q|} < c_1$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} > 0$ and $p_{\mathbf{1}} < 0$.

(84) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} > 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} \geqslant c_1$ and $(q_2)_{\mathbf{2}} > 0$ and $\frac{(q_2)_{\mathbf{1}}}{|q_2|} \geqslant c_1$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1$-FanMorphN(q_1) and $p_2 = c_1$-FanMorphN(q_2), then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(85) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} > 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < c_1$ and $(q_2)_{\mathbf{2}} > 0$ and $\frac{(q_2)_{\mathbf{1}}}{|q_2|} < c_1$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1$-FanMorphN(q_1) and $p_2 = c_1$-FanMorphN(q_2), then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(86) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} > 0$ and $(q_2)_{\mathbf{2}} > 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p_1 = c_1$-FanMorphN(q_1) and $p_2 = c_1$-FanMorphN(q_2), then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(87) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} > 0$ and $\frac{q_{\mathbf{1}}}{|q|} = c_1$. Let p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p = c_1$-FanMorphN(q), then $p_{\mathbf{2}} > 0$ and $p_{\mathbf{1}} = 0$.

(88) For every real number c_1 holds $0_{\mathcal{E}^2_{\mathrm{T}}} = c_1$-FanMorphN$(0_{\mathcal{E}^2_{\mathrm{T}}})$.

4. Fan Morphism for East

Let s be a real number and let q be a point of $\mathcal{E}^2_{\mathrm{T}}$. The functor FanE$(s,q)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 6)
$$\mathrm{FanE}(s,q) = \begin{cases} |q| \cdot [\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1-s})^{\mathbf{2}}}, \frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1-s}], & \text{if } \frac{q_{\mathbf{2}}}{|q|} \geqslant s \text{ and } q_{\mathbf{1}} > 0, \\ |q| \cdot [\sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1+s})^{\mathbf{2}}}, \frac{\frac{q_{\mathbf{2}}}{|q|} - s}{1+s}], & \text{if } \frac{q_{\mathbf{2}}}{|q|} < s \text{ and } q_{\mathbf{1}} > 0, \\ q, & \text{otherwise.} \end{cases}$$

Let s be a real number. The functor s-FanMorphE yielding a function from the carrier of $\mathcal{E}^2_{\mathrm{T}}$ into the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 7) For every point q of $\mathcal{E}^2_{\mathrm{T}}$ holds s-FanMorphE$(q) = $ FanE(s,q).

Next we state a number of propositions:

(89) Let s_1 be a real number. Then

(i) if $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q_{\mathbf{1}} > 0$, then s_1-FanMorphE$(q) = [|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1-s_1})^{\mathbf{2}}}$, $|q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1-s_1}]$, and

(ii) if $q_{\mathbf{1}} \leqslant 0$, then s_1-FanMorphE$(q) = q$.

(90) For every real number s_1 such that $\frac{q_{\mathbf{2}}}{|q|} \leqslant s_1$ and $q_{\mathbf{1}} > 0$ holds s_1-FanMorphE$(q) = [|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1+s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1+s_1}]$.

(91) Let s_1 be a real number such that $-1 < s_1$ and $s_1 < 1$. Then

(i) if $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q_{\mathbf{1}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then s_1-FanMorphE$(q) = [|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1-s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1-s_1}]$, and

(ii) if $\frac{q_{\mathbf{2}}}{|q|} \leqslant s_1$ and $q_{\mathbf{1}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$, then s_1-FanMorphE$(q) = [|q| \cdot \sqrt{1 - (\frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1+s_1})^{\mathbf{2}}}, |q| \cdot \frac{\frac{q_{\mathbf{2}}}{|q|} - s_1}{1+s_1}]$.

(92) Let s_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1-s_1}$, and

(iv) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then f is continuous.

(93) Let s_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1+s_1}$, and

(iv) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \geqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then f is continuous.

(94) Let s_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,

(ii) $s_1 < 1$,

(iii) for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $f(p) = |p| \cdot \sqrt{1 - (\frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1-s_1})^{\mathbf{2}}}$, and

(iv) for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ holds $q_{\mathbf{1}} \geqslant 0$ and $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$ and $q \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$.

Then f is continuous.

(95) Let s_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and f be a map from $(\mathcal{E}^2_{\mathrm{T}}) \restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that

(i) $-1 < s_1$,
(ii) $s_1 < 1$,
(iii) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = |p| \cdot \sqrt{1 - (\frac{\frac{p_{\mathbf{2}}}{|p|} - s_1}{1+s_1})^{\mathbf{2}}}$, and
(iv) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{1}} \geqslant 0$ and $\frac{q_{\mathbf{2}}}{|q|} \leqslant s_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.

Then f is continuous.

(96) Let s_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\mathbf{1}} \geqslant 0 \ \wedge \ q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{2}}}{|p|} \geqslant s_1 \ \wedge \ p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(97) Let s_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\mathbf{1}} \geqslant 0 \ \wedge \ q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\mathbf{2}}}{|p|} \leqslant s_1 \ \wedge \ p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(98) For every real number s_1 and for every subset K_3 of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\mathbf{2}} \geqslant s_1 \cdot |p| \ \wedge \ p_{\mathbf{1}} \geqslant 0\}$ holds K_3 is closed.

(99) For every real number s_1 and for every subset K_3 of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\mathbf{2}} \leqslant s_1 \cdot |p| \ \wedge \ p_{\mathbf{1}} \geqslant 0\}$ holds K_3 is closed.

(100) Let s_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(101) Let s_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(102) Let s_1 be a real number, B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(103) Let s_1 be a real number, B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < s_1$ and $s_1 < 1$ and $f = s_1$-FanMorphE$\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}$) $\setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(104) For every real number s_1 and for every point p of $\mathcal{E}^2_{\rm T}$ holds $|s_1$-FanMorphE$(p)| = |p|$.

(105) For every real number s_1 and for all sets x, K_0 such that $-1 < s_1$ and $s_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{1}} \geqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds s_1-FanMorphE$(x) \in K_0$.

(106) For every real number s_1 and for all sets x, K_0 such that $-1 < s_1$ and $s_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\mathbf{1}} \leqslant 0 \ \wedge \ p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds s_1-FanMorphE$(x) \in K_0$.

(107) Let s_1 be a real number and D be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $D^{\rm c} = \{0_{\mathcal{E}^2_{\rm T}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\rm T}) \restriction D$ into $(\mathcal{E}^2_{\rm T}) \restriction D$ such that $h = s_1$-FanMorphE $\restriction D$ and h is continuous.

(108) Let s_1 be a real number. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a map h from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $h = s_1$-FanMorphE and h is continuous.

(109) For every real number s_1 such that $-1 < s_1$ and $s_1 < 1$ holds s_1-FanMorphE is one-to-one.

(110) For every real number s_1 such that $-1 < s_1$ and $s_1 < 1$ holds s_1-FanMorphE is a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and rng(s_1-FanMorphE) = the carrier of $\mathcal{E}^2_{\rm T}$.

(111) Let s_1 be a real number and p_2 be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a non empty compact subset K of $\mathcal{E}^2_{\rm T}$ such that $K = s_1$-FanMorphE$^\circ K$ and there exists a subset V_2 of $\mathcal{E}^2_{\rm T}$ such that $p_2 \in V_2$ and V_2 is open and $V_2 \subseteq K$ and s_1-FanMorphE$(p_2) \in V_2$.

(112) Let s_1 be a real number. Suppose $-1 < s_1$ and $s_1 < 1$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $f = s_1$-FanMorphE and f is a homeomorphism.

(113) Let s_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\mathbf{1}} > 0$ and $\frac{q_{\mathbf{2}}}{|q|} \geqslant s_1$. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p = s_1$-FanMorphE(q), then $p_{\mathbf{1}} > 0$ and $p_{\mathbf{2}} \geqslant 0$.

(114) Let s_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\mathbf{1}} > 0$ and $\frac{q_{\mathbf{2}}}{|q|} < s_1$. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p = s_1$-FanMorphE(q), then $p_{\mathbf{1}} > 0$ and $p_{\mathbf{2}} < 0$.

(115) Let s_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} > 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} \geqslant s_1$ and $(q_2)_{\mathbf{1}} > 0$ and $\frac{(q_2)_{\mathbf{2}}}{|q_2|} \geqslant s_1$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 = s_1$-FanMorphE(q_1) and $p_2 = s_1$-FanMorphE(q_2), then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(116) Let s_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\mathbf{1}} > 0$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < s_1$ and $(q_2)_{\mathbf{1}} > 0$ and $\frac{(q_2)_{\mathbf{2}}}{|q_2|} < s_1$ and $\frac{(q_1)_{\mathbf{2}}}{|q_1|} < \frac{(q_2)_{\mathbf{2}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 = s_1$-FanMorphE(q_1) and $p_2 = s_1$-FanMorphE(q_2), then $\frac{(p_1)_{\mathbf{2}}}{|p_1|} < \frac{(p_2)_{\mathbf{2}}}{|p_2|}$.

(117) Let s_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $(q_1)_{\bf 1} > 0$ and $(q_2)_{\bf 1} > 0$ and $\frac{(q_1)_{\bf 2}}{|q_1|} < \frac{(q_2)_{\bf 2}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 = s_1$-FanMorphE(q_1) and $p_2 = s_1$-FanMorphE(q_2), then $\frac{(p_1)_{\bf 2}}{|p_1|} < \frac{(p_2)_{\bf 2}}{|p_2|}$.

(118) Let s_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < s_1$ and $s_1 < 1$ and $q_{\bf 1} > 0$ and $\frac{q_{\bf 2}}{|q|} = s_1$. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p = s_1$-FanMorphE(q), then $p_{\bf 1} > 0$ and $p_{\bf 2} = 0$.

(119) For every real number s_1 holds $0_{\mathcal{E}^2_{\rm T}} = s_1$-FanMorphE$(0_{\mathcal{E}^2_{\rm T}})$.

5. Fan Morphism for South

Let s be a real number and let q be a point of $\mathcal{E}^2_{\rm T}$. The functor FanS(s, q) yields a point of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 8) $$\mathrm{FanS}(s,q) = \begin{cases} |q| \cdot [\frac{\frac{q_{\bf 1}}{|q|}-s}{1-s}, -\sqrt{1-(\frac{\frac{q_{\bf 1}}{|q|}-s}{1-s})^{\bf 2}}], & \text{if } \frac{q_{\bf 1}}{|q|} \geqslant s \text{ and } q_{\bf 2} < 0, \\ |q| \cdot [\frac{\frac{q_{\bf 1}}{|q|}-s}{1+s}, -\sqrt{1-(\frac{\frac{q_{\bf 1}}{|q|}-s}{1+s})^{\bf 2}}], & \text{if } \frac{q_{\bf 1}}{|q|} < s \text{ and } q_{\bf 2} < 0, \\ q, & \text{otherwise.} \end{cases}$$

Let c be a real number. The functor c-FanMorphS yielding a function from the carrier of $\mathcal{E}^2_{\rm T}$ into the carrier of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 9) For every point q of $\mathcal{E}^2_{\rm T}$ holds c-FanMorphS$(q) = $ FanS(c, q).

One can prove the following propositions:

(120) Let c_1 be a real number. Then

(i) if $\frac{q_{\bf 1}}{|q|} \geqslant c_1$ and $q_{\bf 2} < 0$, then c_1-FanMorphS$(q) = [|q| \cdot \frac{\frac{q_{\bf 1}}{|q|}-c_1}{1-c_1}, |q| \cdot -\sqrt{1-(\frac{\frac{q_{\bf 1}}{|q|}-c_1}{1-c_1})^{\bf 2}}]$, and

(ii) if $q_{\bf 2} \geqslant 0$, then c_1-FanMorphS$(q) = q$.

(121) For every real number c_1 such that $\frac{q_{\bf 1}}{|q|} \leqslant c_1$ and $q_{\bf 2} < 0$ holds c_1-FanMorphS$(q) = [|q| \cdot \frac{\frac{q_{\bf 1}}{|q|}-c_1}{1+c_1}, |q| \cdot -\sqrt{1-(\frac{\frac{q_{\bf 1}}{|q|}-c_1}{1+c_1})^{\bf 2}}]$.

(122) Let c_1 be a real number such that $-1 < c_1$ and $c_1 < 1$. Then

(i) if $\frac{q_{\bf 1}}{|q|} \geqslant c_1$ and $q_{\bf 2} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$, then c_1-FanMorphS$(q) = [|q| \cdot \frac{\frac{q_{\bf 1}}{|q|}-c_1}{1-c_1}, |q| \cdot -\sqrt{1-(\frac{\frac{q_{\bf 1}}{|q|}-c_1}{1-c_1})^{\bf 2}}]$, and

(ii) if $\frac{q_{\bf 1}}{|q|} \leqslant c_1$ and $q_{\bf 2} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$, then c_1-FanMorphS$(q) = [|q| \cdot \frac{\frac{q_{\bf 1}}{|q|}-c_1}{1+c_1}, |q| \cdot -\sqrt{1-(\frac{\frac{q_{\bf 1}}{|q|}-c_1}{1+c_1})^{\bf 2}}]$.

(123) Let c_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
  (i) $-1 < c_1$,
  (ii) $c_1 < 1$,
  (iii) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 - c_1}$, and
  (iv) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.
  Then f is continuous.

(124) Let c_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
  (i) $-1 < c_1$,
  (ii) $c_1 < 1$,
  (iii) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = |p| \cdot \frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 + c_1}$, and
  (iv) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \leqslant 0$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.
  Then f is continuous.

(125) Let c_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
  (i) $-1 < c_1$,
  (ii) $c_1 < 1$,
  (iii) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = |p| \cdot -\sqrt{1 - (\frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 - c_1})^{\mathbf{2}}}$, and
  (iv) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \leqslant 0$ and $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.
  Then f is continuous.

(126) Let c_1 be a real number, K_1 be a non empty subset of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_1$ into $\mathbb{R}^{\mathbf{1}}$. Suppose that
  (i) $-1 < c_1$,
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  (iii) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $f(p) = |p| \cdot -\sqrt{1 - (\frac{\frac{p_{\mathbf{1}}}{|p|} - c_1}{1 + c_1})^{\mathbf{2}}}$, and
  (iv) for every point q of $\mathcal{E}^2_{\rm T}$ such that $q \in$ the carrier of $(\mathcal{E}^2_{\rm T})\restriction K_1$ holds $q_{\mathbf{2}} \leqslant 0$ and $\frac{q_{\mathbf{1}}}{|q|} \leqslant c_1$ and $q \neq 0_{\mathcal{E}^2_{\rm T}}$.
  Then f is continuous.

(127) Let c_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f =$

c_1-FanMorphS $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\bf 2} \leqslant 0 \wedge q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\bf 1}}{|p|} \geqslant c_1 \wedge p_{\bf 2} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(128) Let c_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphS $\restriction K_0$ and $B_0 = \{q; q$ ranges over points of $\mathcal{E}^2_{\rm T}$: $q_{\bf 2} \leqslant 0 \wedge q \neq 0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : \frac{p_{\bf 1}}{|p|} \leqslant c_1 \wedge p_{\bf 2} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(129) For every real number c_1 and for every subset K_3 of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\bf 1} \geqslant c_1 \cdot |p| \wedge p_{\bf 2} \leqslant 0\}$ holds K_3 is closed.

(130) For every real number c_1 and for every subset K_3 of $\mathcal{E}^2_{\rm T}$ such that $K_3 = \{p : p_{\bf 1} \leqslant c_1 \cdot |p| \wedge p_{\bf 2} \leqslant 0\}$ holds K_3 is closed.

(131) Let c_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphS $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}) \setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\bf 2} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(132) Let c_1 be a real number, K_0, B_0 be subsets of $\mathcal{E}^2_{\rm T}$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphS $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}) \setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\bf 2} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

(133) Let c_1 be a real number, B_0 be a subset of $\mathcal{E}^2_{\rm T}$, K_0 be a subset of $(\mathcal{E}^2_{\rm T})\restriction B_0$, and f be a map from $(\mathcal{E}^2_{\rm T})\restriction B_0\restriction K_0$ into $(\mathcal{E}^2_{\rm T})\restriction B_0$. Suppose $-1 < c_1$ and $c_1 < 1$ and $f = c_1$-FanMorphS $\restriction K_0$ and $B_0 =$ (the carrier of $\mathcal{E}^2_{\rm T}) \setminus \{0_{\mathcal{E}^2_{\rm T}}\}$ and $K_0 = \{p : p_{\bf 2} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$. Then f is continuous.

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(135) For every real number c_1 and for every point p of $\mathcal{E}^2_{\rm T}$ holds $|c_1$-FanMorphS$(p)| = |p|$.

(136) For every real number c_1 and for all sets x, K_0 such that $-1 < c_1$ and $c_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\bf 2} \leqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds c_1-FanMorphS$(x) \in K_0$.

(137) For every real number c_1 and for all sets x, K_0 such that $-1 < c_1$ and $c_1 < 1$ and $x \in K_0$ and $K_0 = \{p : p_{\bf 2} \geqslant 0 \wedge p \neq 0_{\mathcal{E}^2_{\rm T}}\}$ holds c_1-FanMorphS$(x) \in K_0$.

(138) Let c_1 be a real number and D be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $D^{\rm c} = \{0_{\mathcal{E}^2_{\rm T}}\}$. Then there exists a map h from $(\mathcal{E}^2_{\rm T})\restriction D$ into $(\mathcal{E}^2_{\rm T})\restriction D$ such that $h = c_1$-FanMorphS $\restriction D$ and h is continuous.

(139) Let c_1 be a real number. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a map h from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $h = c_1$-FanMorphS and h is continuous.

(140) For every real number c_1 such that $-1 < c_1$ and $c_1 < 1$ holds c_1-FanMorphS is one-to-one.

(141) For every real number c_1 such that $-1 < c_1$ and $c_1 < 1$ holds c_1-FanMorphS is a map from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ and $\mathrm{rng}(c_1\text{-FanMorphS})$ = the carrier of $\mathcal{E}^2_{\rm T}$.

(142) Let c_1 be a real number and p_2 be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a non empty compact subset K of $\mathcal{E}^2_{\rm T}$ such that $K = c_1\text{-FanMorphS}^\circ K$ and there exists a subset V_2 of $\mathcal{E}^2_{\rm T}$ such that $p_2 \in V_2$ and V_2 is open and $V_2 \subseteq K$ and $c_1\text{-FanMorphS}(p_2) \in V_2$.

(143) Let c_1 be a real number. Suppose $-1 < c_1$ and $c_1 < 1$. Then there exists a map f from $\mathcal{E}^2_{\rm T}$ into $\mathcal{E}^2_{\rm T}$ such that $f = c_1$-FanMorphS and f is a homeomorphism.

(144) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$ and $\frac{q_{\mathbf{1}}}{|q|} \geqslant c_1$. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p = c_1\text{-FanMorphS}(q)$, then $p_{\mathbf{2}} < 0$ and $p_{\mathbf{1}} \geqslant 0$.

(145) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$ and $\frac{q_{\mathbf{1}}}{|q|} < c_1$. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p = c_1\text{-FanMorphS}(q)$, then $p_{\mathbf{2}} < 0$ and $p_{\mathbf{1}} < 0$.

(146) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} < 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} \geqslant c_1$ and $(q_2)_{\mathbf{2}} < 0$ and $\frac{(q_2)_{\mathbf{1}}}{|q_2|} \geqslant c_1$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 = c_1\text{-FanMorphS}(q_1)$ and $p_2 = c_1\text{-FanMorphS}(q_2)$, then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(147) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} < 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < c_1$ and $(q_2)_{\mathbf{2}} < 0$ and $\frac{(q_2)_{\mathbf{1}}}{|q_2|} < c_1$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 = c_1\text{-FanMorphS}(q_1)$ and $p_2 = c_1\text{-FanMorphS}(q_2)$, then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(148) Let c_1 be a real number and q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $(q_1)_{\mathbf{2}} < 0$ and $(q_2)_{\mathbf{2}} < 0$ and $\frac{(q_1)_{\mathbf{1}}}{|q_1|} < \frac{(q_2)_{\mathbf{1}}}{|q_2|}$. Let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If $p_1 = c_1\text{-FanMorphS}(q_1)$ and $p_2 = c_1\text{-FanMorphS}(q_2)$, then $\frac{(p_1)_{\mathbf{1}}}{|p_1|} < \frac{(p_2)_{\mathbf{1}}}{|p_2|}$.

(149) Let c_1 be a real number and q be a point of $\mathcal{E}^2_{\rm T}$. Suppose $-1 < c_1$ and $c_1 < 1$ and $q_{\mathbf{2}} < 0$ and $\frac{q_{\mathbf{1}}}{|q|} = c_1$. Let p be a point of $\mathcal{E}^2_{\rm T}$. If $p = c_1\text{-FanMorphS}(q)$, then $p_{\mathbf{2}} < 0$ and $p_{\mathbf{1}} = 0$.

(150) For every real number c_1 holds $0_{\mathcal{E}^2_{\rm T}} = c_1\text{-FanMorphS}(0_{\mathcal{E}^2_{\rm T}})$.

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