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On the General Position of Special Polygons[1]

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Summary. In this paper we introduce the notion of general position. We also show some auxiliary theorems for proving Jordan curve theorem. The following main theorems are proved:

1. End points of a polygon are in the same component of a complement of another polygon if number of common points of these polygons is even;
2. Two points of polygon L are in the same component of a complement of polygon M if two points of polygon M are in the same component of polygon L.

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The papers [23], [6], [26], [20], [2], [18], [22], [16], [27], [1], [8], [5], [3], [25], [11], [4], [21], [19], [9], [10], [14], [15], [12], [13], [17], [24], and [7] provide the terminology and notation for this paper.

1. Preliminaries

We adopt the following rules: i, j, k, n denote natural numbers, a, b, c, x denote sets, and r denotes a real number.

The following four propositions are true:

(1) If $1 < i$, then $0 < i -' 1$.

(2) If $1 \leqslant i$, then $i -' 1 < i$.

(3) 1 is odd.

(4) Let given n, f be a finite sequence of elements of $\mathcal{E}^n_{\rm T}$, and given i. If $1 \leqslant i$ and $i + 1 \leqslant \operatorname{len} f$, then $f_i \in \operatorname{rng} f$ and $f_{i+1} \in \operatorname{rng} f$.

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Let us mention that every finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ which is s.n.c. is also s.c.c..

Next we state two propositions:

(5) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f \frown g$ is unfolded and s.c.c. and $\operatorname{len} g \geqslant 2$, then f is unfolded and s.n.c..

(6) For all finite sequences g_1, g_2 of elements of $\mathcal{E}^2_{\mathrm{T}}$ holds $\widetilde{\mathcal{L}}(g_1) \subseteq \widetilde{\mathcal{L}}(g_1 \frown g_2)$.

2. The Notion of General Position and Its Properties

Let us consider n and let f_1, f_2 be finite sequences of elements of $\mathcal{E}^n_{\mathrm{T}}$. We say that f_1 is in general position wrt f_2 if and only if:

(Def. 1) $\widetilde{\mathcal{L}}(f_1)$ misses $\operatorname{rng} f_2$ and for every i such that $1 \leqslant i$ and $i < \operatorname{len} f_2$ holds $\widetilde{\mathcal{L}}(f_1) \cap \mathcal{L}(f_2, i)$ is trivial.

Let us consider n and let f_1, f_2 be finite sequences of elements of $\mathcal{E}^n_{\mathrm{T}}$. We say that f_1 and f_2 are in general position if and only if:

(Def. 2) f_1 is in general position wrt f_2 and f_2 is in general position wrt f_1.

Let us note that the predicate f_1 and f_2 are in general position is symmetric.

The following propositions are true:

(7) Let f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f_1 and f_2 are in general position. Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $f = f_2 \restriction \operatorname{Seg} k$, then f_1 and f are in general position.

(8) Let f_1, f_2, g_1, g_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $f_1 \frown f_2$ and $g_1 \frown g_2$ are in general position. Then $f_1 \frown f_2$ and g_1 are in general position.

In the sequel f, g are finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$.

The following propositions are true:

(9) For all k, f, g such that $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} g$ and f and g are in general position holds $g(k) \in (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$ and $g(k+1) \in (\widetilde{\mathcal{L}}(f))^{\mathrm{c}}$.

(10) Let f_1, f_2 be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f_1 and f_2 are in general position. Let given i, j. If $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f_1$ and $1 \leqslant j$ and $j+1 \leqslant \operatorname{len} f_2$, then $\mathcal{L}(f_1, i) \cap \mathcal{L}(f_2, j)$ is trivial.

(11) For all f, g holds $\{\mathcal{L}(f,i) : 1 \leqslant i \ \wedge \ i+1 \leqslant \operatorname{len} f\} \Cap \{\mathcal{L}(g,j) : 1 \leqslant j \ \wedge \ j+1 \leqslant \operatorname{len} g\}$ is finite.

(12) For all f, g such that f and g are in general position holds $\widetilde{\mathcal{L}}(f) \cap \widetilde{\mathcal{L}}(g)$ is finite.

(13) For all f, g such that f and g are in general position and for every k holds $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(g, k)$ is finite.

3. Properties of Being in the Same Component of a Complement of a Polygon

We use the following convention: f is a non constant standard special circular sequence, g is a special finite sequence of elements of $\mathcal{E}^2_{\rm T}$, and p, p_1, p_2, q are points of $\mathcal{E}^2_{\rm T}$.

One can prove the following propositions:

(14) For all f, p_1, p_2 such that $\mathcal{L}(p_1,p_2)$ misses $\widetilde{\mathcal{L}}(f)$ there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $p_1 \in C$ and $p_2 \in C$.

(15) There exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $a \in C$ and $b \in C$ if and only if $a \in \mathrm{RightComp}(f)$ and $b \in \mathrm{RightComp}(f)$ or $a \in \mathrm{LeftComp}(f)$ and $b \in \mathrm{LeftComp}(f)$.

(16) $a \in (\widetilde{\mathcal{L}}(f))^{\rm c}$ and $b \in (\widetilde{\mathcal{L}}(f))^{\rm c}$ and it is not true that there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $a \in C$ and $b \in C$ if and only if $a \in \mathrm{LeftComp}(f)$ and $b \in \mathrm{RightComp}(f)$ or $a \in \mathrm{RightComp}(f)$ and $b \in \mathrm{LeftComp}(f)$.

(17) Let given f, a, b, c. Suppose that
(i) there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $a \in C$ and $b \in C$, and
(ii) there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $b \in C$ and $c \in C$.
Then there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $a \in C$ and $c \in C$.

(18) Let given f, a, b, c. Suppose that
(i) $a \in (\widetilde{\mathcal{L}}(f))^{\rm c}$,
(ii) $b \in (\widetilde{\mathcal{L}}(f))^{\rm c}$,
(iii) $c \in (\widetilde{\mathcal{L}}(f))^{\rm c}$,
(iv) it is not true that there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $a \in C$ and $b \in C$, and
(v) it is not true that there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $b \in C$ and $c \in C$.
Then there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $a \in C$ and $c \in C$.

4. Cells Are Convex

In the sequel G denotes a Go-board.

One can prove the following propositions:

(19) If $i \leqslant \mathrm{len}\, G$, then $\mathrm{vstrip}(G,i)$ is convex.

(20) If $j \leqslant \mathrm{width}\, G$, then $\mathrm{hstrip}(G,j)$ is convex.

(21) If $i \leqslant \operatorname{len} G$ and $j \leqslant \operatorname{width} G$, then $\operatorname{cell}(G, i, j)$ is convex.

(22) For all f, k such that $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} f$ holds $\operatorname{leftcell}(f, k)$ is convex.

(23) For all f, k such that $1 \leqslant k$ and $k + 1 \leqslant \operatorname{len} f$ holds $\operatorname{left\_cell}(f, k, \text{the Go-board of } f)$ is convex and $\operatorname{right\_cell}(f, k, \text{the Go-board of } f)$ is convex.

5. Properties of Points Lying on the Same Line

The following propositions are true:

(24) Let given p_1, p_2, f and r be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $r \in \mathcal{L}(p_1, p_2)$ and there exists x such that $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(p_1, p_2) = \{x\}$ and $r \notin \widetilde{\mathcal{L}}(f)$. Then $\widetilde{\mathcal{L}}(f)$ misses $\mathcal{L}(p_1, r)$ or $\widetilde{\mathcal{L}}(f)$ misses $\mathcal{L}(r, p_2)$.

(25) For all points p, q, r, s of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathcal{L}(p, q)$ is vertical and $\mathcal{L}(r, s)$ is vertical and $\mathcal{L}(p, q)$ meets $\mathcal{L}(r, s)$ holds $p_{\mathbf{1}} = r_{\mathbf{1}}$.

(26) For all p, p_1, p_2 such that $p \notin \mathcal{L}(p_1, p_2)$ and $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} = p_{\mathbf{2}}$ holds $p_1 \in \mathcal{L}(p, p_2)$ or $p_2 \in \mathcal{L}(p, p_1)$.

(27) For all p, p_1, p_2 such that $p \notin \mathcal{L}(p_1, p_2)$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} = p_{\mathbf{1}}$ holds $p_1 \in \mathcal{L}(p, p_2)$ or $p_2 \in \mathcal{L}(p, p_1)$.

(28) If $p \neq p_1$ and $p \neq p_2$ and $p \in \mathcal{L}(p_1, p_2)$, then $p_1 \notin \mathcal{L}(p, p_2)$.

(29) Let given p, p_1, p_2, q. Suppose $q \notin \mathcal{L}(p_1, p_2)$ and $p \in \mathcal{L}(p_1, p_2)$ and $p \neq p_1$ and $p \neq p_2$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_2)_{\mathbf{1}} = q_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_2)_{\mathbf{2}} = q_{\mathbf{2}}$. Then $p_1 \in \mathcal{L}(q, p)$ or $p_2 \in \mathcal{L}(q, p)$.

(30) Let p_1, p_2, p_3, p_4, p be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_3)_{\mathbf{1}} = (p_4)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_3)_{\mathbf{2}} = (p_4)_{\mathbf{2}}$ but $\mathcal{L}(p_1, p_2) \cap \mathcal{L}(p_3, p_4) = \{p\}$. Then $p = p_1$ or $p = p_2$ or $p = p_3$.

6. The Position of the Points of a Polygon with Respect to Another Polygon

We now state several propositions:

(31) Let given p, p_1, p_2, f. Suppose $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(p_1, p_2) = \{p\}$. Let r be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) $r \notin \mathcal{L}(p_1, p_2)$,

(ii) $p_1 \notin \widetilde{\mathcal{L}}(f)$,

(iii) $p_2 \notin \widetilde{\mathcal{L}}(f)$,

(iv) $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} = r_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} = r_{\mathbf{2}}$,

(v) there exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $r \in \operatorname{right\_cell}(f, i, \text{the Go-board of } f)$ or $r \in \operatorname{left\_cell}(f, i, \text{the Go-board of } f)$ and $p \in \mathcal{L}(f, i)$, and

(vi) $r \notin \widetilde{\mathcal{L}}(f)$.

Then

(vii) there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $r \in C$ and $p_1 \in C$, or

(viii) there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $r \in C$ and $p_2 \in C$.

(32) Let given f, p_1, p_2, p. Suppose $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(p_1, p_2) = \{p\}$. Let r_1, r_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose that

(i) $p_1 \notin \widetilde{\mathcal{L}}(f)$,

(ii) $p_2 \notin \widetilde{\mathcal{L}}(f)$,

(iii) $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $(p_1)_{\mathbf{1}} = (r_1)_{\mathbf{1}}$ and $(r_1)_{\mathbf{1}} = (r_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $(p_1)_{\mathbf{2}} = (r_1)_{\mathbf{2}}$ and $(r_1)_{\mathbf{2}} = (r_2)_{\mathbf{2}}$,

(iv) there exists i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} f$ and $r_1 \in \text{left\_cell}(f, i, \text{the Go-board of } f)$ and $r_2 \in \text{right\_cell}(f, i, \text{the Go-board of } f)$ and $p \in \mathcal{L}(f, i)$,

(v) $r_1 \notin \widetilde{\mathcal{L}}(f)$, and

(vi) $r_2 \notin \widetilde{\mathcal{L}}(f)$.

Then it is not true that there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $p_1 \in C$ and $p_2 \in C$.

(33) Let given p, f, p_1, p_2. Suppose $\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(p_1, p_2) = \{p\}$ and $(p_1)_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $p_1 \notin \widetilde{\mathcal{L}}(f)$ and $p_2 \notin \widetilde{\mathcal{L}}(f)$ and $\operatorname{rng} f$ misses $\mathcal{L}(p_1, p_2)$. Then it is not true that there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $p_1 \in C$ and $p_2 \in C$.

(34) Let f be a non constant standard special circular sequence and g be a special finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f and g are in general position. Let given k. Suppose $1 \leqslant k$ and $k+1 \leqslant \operatorname{len} g$. Then $\overline{\overline{\widetilde{\mathcal{L}}(f) \cap \mathcal{L}(g, k)}}$ is an even natural number if and only if there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f))^{\rm c}$ and $g(k) \in C$ and $g(k+1) \in C$.

(35) Let f_1, f_2, g_1 be special finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose that

(i) $f_1 \frown f_2$ is a non constant standard special circular sequence,

(ii) $f_1 \frown f_2$ and g_1 are in general position,

(iii) $\operatorname{len} g_1 \geqslant 2$, and

(iv) g_1 is unfolded and s.n.c..

Then $\overline{\overline{\widetilde{\mathcal{L}}(f_1 \frown f_2) \cap \widetilde{\mathcal{L}}(g_1)}}$ is an even natural number if and only if there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f_1 \frown f_2))^{\rm c}$ and $g_1(1) \in C$ and $g_1(\operatorname{len} g_1) \in C$.

(36) Let f_1, f_2, g_1, g_2 be special finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose that

(i) $f_1 \frown f_2$ is a non constant standard special circular sequence,

(ii) $g_1 \frown g_2$ is a non constant standard special circular sequence,

(iii) $\widetilde{\mathcal{L}}(f_1)$ misses $\widetilde{\mathcal{L}}(g_2)$,

(iv) $\widetilde{\mathcal{L}}(f_2)$ misses $\widetilde{\mathcal{L}}(g_1)$, and

(v) $f_1 \frown\frown f_2$ and $g_1 \frown\frown g_2$ are in general position.

Let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose that $f_1(1) = p_1$ and $f_1(\mathrm{len}\, f_1) = p_2$ and $g_1(1) = q_1$ and $g_1(\mathrm{len}\, g_1) = q_2$ and $(f_1)_{\mathrm{len}\, f_1} = (f_2)_1$ and $(g_1)_{\mathrm{len}\, g_1} = (g_2)_1$ and $p_1 \neq p_2$ and $q_1 \neq q_2$ and $p_1 \in \widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2)$ and $q_1 \in \widetilde{\mathcal{L}}(g_1) \cap \widetilde{\mathcal{L}}(g_2)$ and there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(f_1 \frown\frown f_2))^{\rm c}$ and $q_1 \in C$ and $q_2 \in C$. Then there exists a subset C of $\mathcal{E}^2_{\rm T}$ such that C is a component of $(\widetilde{\mathcal{L}}(g_1 \frown\frown g_2))^{\rm c}$ and $p_1 \in C$ and $p_2 \in C$.

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