

Properties of the Upper and Lower Sequence on the Cage¹

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The terminology and notation used here are introduced in the following articles: [24], [27], [1], [3], [4], [2], [14], [12], [25], [22], [23], [11], [21], [8], [9], [6], [26], [15], [10], [18], [17], [5], [20], [19], [7], [13], and [16].

In this paper n is a natural number.

We now state a number of propositions:

- (1) For all subsets A, B of \mathcal{E}_T^2 such that A meets B holds $\text{proj}1^\circ A$ meets $\text{proj}1^\circ B$.
- (2) Let A, B be subsets of \mathcal{E}_T^2 and s be a real number. If A misses B and $A \subseteq \text{HorizontalLine } s$ and $B \subseteq \text{HorizontalLine } s$, then $\text{proj}1^\circ A$ misses $\text{proj}1^\circ B$.
- (3) For every closed subset S of \mathcal{E}_T^2 such that S is Bounded holds $\text{proj}1^\circ S$ is closed.
- (4) For every compact subset S of \mathcal{E}_T^2 holds $\text{proj}1^\circ S$ is compact.
- (5) Let p, q, p_1, q_1 be points of \mathcal{E}_T^2 . Suppose $\mathcal{L}(p, q)$ is vertical and $\mathcal{L}(p_1, q_1)$ is vertical and $p_1 = (p_1)_1$ and $p_2 \leq (p_1)_2$ and $(p_1)_2 \leq (q_1)_2$ and $(q_1)_2 \leq q_2$. Then $\mathcal{L}(p_1, q_1) \subseteq \mathcal{L}(p, q)$.
- (6) Let p, q, p_1, q_1 be points of \mathcal{E}_T^2 . Suppose $\mathcal{L}(p, q)$ is horizontal and $\mathcal{L}(p_1, q_1)$ is horizontal and $p_2 = (p_1)_2$ and $p_1 \leq (p_1)_1$ and $(p_1)_1 \leq (q_1)_1$ and $(q_1)_1 \leq q_1$. Then $\mathcal{L}(p_1, q_1) \subseteq \mathcal{L}(p, q)$.
- (7) Let G be a Go-board and i, j, k, j_1, k_1 be natural numbers. Suppose $1 \leq i$ and $i \leq \text{len } G$ and $1 \leq j$ and $j \leq j_1$ and $j_1 \leq k_1$ and $k_1 \leq k$ and $k \leq \text{width } G$. Then $\mathcal{L}(G \circ (i, j_1), G \circ (i, k_1)) \subseteq \mathcal{L}(G \circ (i, j), G \circ (i, k))$.

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- (8) Let G be a Go-board and i, j, k, j_1, k_1 be natural numbers. Suppose $1 \leq i$ and $i \leq \text{width } G$ and $1 \leq j$ and $j \leq j_1$ and $j_1 \leq k_1$ and $k_1 \leq k$ and $k \leq \text{len } G$. Then $\mathcal{L}(G \circ (j_1, i), G \circ (k_1, i)) \subseteq \mathcal{L}(G \circ (j, i), G \circ (k, i))$.
- (9) Let G be a Go-board and j, k, j_1, k_1 be natural numbers. Suppose $1 \leq j$ and $j \leq j_1$ and $j_1 \leq k_1$ and $k_1 \leq k$ and $k \leq \text{width } G$. Then $\mathcal{L}(G \circ (\text{Center } G, j_1), G \circ (\text{Center } G, k_1)) \subseteq \mathcal{L}(G \circ (\text{Center } G, j), G \circ (\text{Center } G, k))$.
- (10) Let G be a Go-board. Suppose $\text{len } G = \text{width } G$. Let j, k, j_1, k_1 be natural numbers. Suppose $1 \leq j$ and $j \leq j_1$ and $j_1 \leq k_1$ and $k_1 \leq k$ and $k \leq \text{len } G$. Then $\mathcal{L}(G \circ (j_1, \text{Center } G), G \circ (k_1, \text{Center } G)) \subseteq \mathcal{L}(G \circ (j, \text{Center } G), G \circ (k, \text{Center } G))$.
- (11) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq i$ and $i \leq \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then there exists a natural number j_1 such that $j \leq j_1$ and $j_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j_1), \text{Gauge}(C, n) \circ (i, k)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, j_1)\}$.
- (12) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq i$ and $i \leq \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, k) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$. Then there exists a natural number k_1 such that $j \leq k_1$ and $k_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k_1)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, k_1)\}$.
- (13) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq i$ and $i \leq \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (i, k) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$. Then there exist natural numbers j_1, k_1 such that $j \leq j_1$ and $j_1 \leq k_1$ and $k_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j_1), \text{Gauge}(C, n) \circ (i, k_1)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, j_1)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j_1), \text{Gauge}(C, n) \circ (i, k_1)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, k_1)\}$.
- (14) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq j$ and $j \leq k$ and $k \leq \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (j, i) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then there exists a natural number j_1 such that $j \leq j_1$ and $j_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j_1, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (j_1, i)\}$.
- (15) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq j$ and $j \leq k$ and $k \leq \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (k, i) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$. Then there exists a natural number k_1 such that $j \leq k_1$ and $k_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k_1, i)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (k_1, i)\}$.

- $\tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (k_1, i)\}.$
- (16) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq j$ and $j \leq k$ and $k \leq \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (j, i) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (k, i) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$. Then there exist natural numbers j_1, k_1 such that $j \leq j_1$ and $j_1 \leq k_1$ and $k_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j_1, i), \text{Gauge}(C, n) \circ (k_1, i)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (j_1, i)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j_1, i), \text{Gauge}(C, n) \circ (k_1, i)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (k_1, i)\}.$
- (17) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq i$ and $i \leq \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$. Then there exists a natural number j_1 such that $j \leq j_1$ and $j_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j_1), \text{Gauge}(C, n) \circ (i, k)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, j_1)\}.$
- (18) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq i$ and $i \leq \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, k) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then there exists a natural number k_1 such that $j \leq k_1$ and $k_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k_1)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, k_1)\}.$
- (19) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq i$ and $i \leq \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (i, k) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then there exist natural numbers j_1, k_1 such that $j \leq j_1$ and $j_1 \leq k_1$ and $k_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j_1), \text{Gauge}(C, n) \circ (i, k_1)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, j_1)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j_1), \text{Gauge}(C, n) \circ (i, k_1)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, k_1)\}.$
- (20) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq j$ and $j \leq k$ and $k \leq \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (j, i) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$. Then there exists a natural number j_1 such that $j \leq j_1$ and $j_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j_1, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (j_1, i)\}.$
- (21) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq j$ and $j \leq k$ and $k \leq \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (k, i) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then there exists a natural number k_1 such that $j \leq k_1$ and $k_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k_1, i)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (k_1, i)\}.$

- (22) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 \leq j$ and $j \leq k$ and $k \leq \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (j, i) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (k, i) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then there exist natural numbers j_1, k_1 such that $j \leq j_1$ and $j_1 \leq k_1$ and $k_1 \leq k$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j_1, i), \text{Gauge}(C, n) \circ (k_1, i)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (j_1, i)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j_1, i), \text{Gauge}(C, n) \circ (k_1, i)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (k_1, i)\}$.
- (23) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, k) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k))$ meets LowerArc C .
- (24) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, k) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k))$ meets UpperArc C .
- (25) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $n > 0$ and $\text{Gauge}(C, n) \circ (i, k) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$ and $\text{Gauge}(C, n) \circ (i, j) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k))$ meets LowerArc C .
- (26) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $n > 0$ and $\text{Gauge}(C, n) \circ (i, k) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$ and $\text{Gauge}(C, n) \circ (i, j) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k))$ meets UpperArc C .
- (27) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), k) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$ and $\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j), \text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), k))$ meets LowerArc C .
- (28) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), k) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$ and $\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j), \text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), k))$ meets LowerArc C .

- (Center Gauge($C, n + 1, k$)) meets UpperArc C .
- (29) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j, k be natural numbers. Suppose $1 < j$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (k, i) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (j, i) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then $j \neq k$.
- (30) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (k, i)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (j, i)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets LowerArc C .
- (31) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (k, i)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (j, i)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets UpperArc C .
- (32) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (k, i) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (j, i) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets LowerArc C .
- (33) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (k, i) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (j, i) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets UpperArc C .
- (34) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $n > 0$ and $\text{Gauge}(C, n) \circ (k, i) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$ and $\text{Gauge}(C, n) \circ (j, i) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets LowerArc C .
- (35) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $n > 0$ and $\text{Gauge}(C, n) \circ (k, i) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$ and $\text{Gauge}(C, n) \circ (j, i) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets UpperArc C .
- (36) Let C be a simple closed curve and j, k be natural numbers. Sup-

pose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (k, \text{Center Gauge}(C, n+1)) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$ and $\text{Gauge}(C, n+1) \circ (j, \text{Center Gauge}(C, n+1)) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (j, \text{Center Gauge}(C, n+1)), \text{Gauge}(C, n+1) \circ (k, \text{Center Gauge}(C, n+1)))$ meets $\text{LowerArc } C$.

- (37) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (k, \text{Center Gauge}(C, n+1)) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$ and $\text{Gauge}(C, n+1) \circ (j, \text{Center Gauge}(C, n+1)) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (j, \text{Center Gauge}(C, n+1)), \text{Gauge}(C, n+1) \circ (k, \text{Center Gauge}(C, n+1)))$ meets $\text{UpperArc } C$.
- (38) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (j, i)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (k, i)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets $\text{LowerArc } C$.
- (39) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (j, i)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (k, i)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets $\text{UpperArc } C$.
- (40) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (j, i) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (k, i) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets $\text{LowerArc } C$.
- (41) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (j, i) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (k, i) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets $\text{UpperArc } C$.
- (42) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$ and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $n > 0$ and $\text{Gauge}(C, n) \circ (j, i) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$ and $\text{Gauge}(C, n) \circ (k, i) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets $\text{LowerArc } C$.
- (43) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n)$

- and $1 \leq i$ and $i \leq \text{width Gauge}(C, n)$ and $n > 0$ and $\text{Gauge}(C, n) \circ (j, i) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$ and $\text{Gauge}(C, n) \circ (k, i) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (j, i), \text{Gauge}(C, n) \circ (k, i))$ meets $\text{UpperArc } C$.
- (44) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n + 1)$ and $\text{Gauge}(C, n + 1) \circ (j, \text{Center Gauge}(C, n + 1)) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n + 1))$ and $\text{Gauge}(C, n + 1) \circ (k, \text{Center Gauge}(C, n + 1)) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n + 1))$. Then $\mathcal{L}(\text{Gauge}(C, n + 1) \circ (j, \text{Center Gauge}(C, n + 1)), \text{Gauge}(C, n + 1) \circ (k, \text{Center Gauge}(C, n + 1)))$ meets $\text{LowerArc } C$.
- (45) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 < j$ and $j \leq k$ and $k < \text{len Gauge}(C, n + 1)$ and $\text{Gauge}(C, n + 1) \circ (j, \text{Center Gauge}(C, n + 1)) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n + 1))$ and $\text{Gauge}(C, n + 1) \circ (k, \text{Center Gauge}(C, n + 1)) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n + 1))$. Then $\mathcal{L}(\text{Gauge}(C, n + 1) \circ (j, \text{Center Gauge}(C, n + 1)), \text{Gauge}(C, n + 1) \circ (k, \text{Center Gauge}(C, n + 1)))$ meets $\text{UpperArc } C$.
- (46) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leq i_2$ and $i_2 < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $(\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_2, k)\}$ and $(\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_1, j)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))$ meets $\text{UpperArc } C$.
- (47) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leq i_2$ and $i_2 < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $(\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_2, k)\}$ and $(\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_1, j)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))$ meets $\text{LowerArc } C$.
- (48) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leq i_1$ and $i_1 < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $(\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_2, k)\}$ and $(\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_1, j)\}$.

Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))$ meets $\text{UpperArc } C$.

- (49) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leq i_1$ and $i_1 < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $(\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_2, k)\}$ and $(\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_1, j)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))$ meets $\text{LowerArc } C$.
- (50) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \text{len Gauge}(C, n+1)$ and $1 < i_2$ and $i_2 < \text{len Gauge}(C, n+1)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (i_1, k) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$ and $\text{Gauge}(C, n+1) \circ (i_2, j) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (i_2, j), \text{Gauge}(C, n+1) \circ (i_2, k)) \cup \mathcal{L}(\text{Gauge}(C, n+1) \circ (i_2, k), \text{Gauge}(C, n+1) \circ (i_1, k))$ meets $\text{UpperArc } C$.
- (51) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \text{len Gauge}(C, n+1)$ and $1 < i_2$ and $i_2 < \text{len Gauge}(C, n+1)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (i_1, k) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$ and $\text{Gauge}(C, n+1) \circ (i_2, j) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (i_2, j), \text{Gauge}(C, n+1) \circ (i_2, k)) \cup \mathcal{L}(\text{Gauge}(C, n+1) \circ (i_2, k), \text{Gauge}(C, n+1) \circ (i_1, k))$ meets $\text{LowerArc } C$.
- (52) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n+1)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (i, k) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$ and $\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j), \text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), k)) \cup \mathcal{L}(\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), k), \text{Gauge}(C, n+1) \circ (i, k))$ meets $\text{UpperArc } C$.
- (53) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n+1)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (i, k) \in \text{UpperArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$ and $\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j) \in \text{LowerArc } \tilde{\mathcal{L}}(\text{Cage}(C, n+1))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j), \text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), k)) \cup \mathcal{L}(\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), k), \text{Gauge}(C, n+1) \circ (i, k))$ meets $\text{LowerArc } C$.

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [5] Czesław Byliński. Gauges. *Formalized Mathematics*, 8(1):25–27, 1999.
- [6] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in \mathcal{E}^2 . *Formalized Mathematics*, 6(3):427–440, 1997.
- [7] Czesław Byliński and Mariusz Żynel. Cages - the external approximation of Jordan's curve. *Formalized Mathematics*, 9(1):19–24, 2001.
- [8] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(2):383–386, 1990.
- [9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.
- [10] Agata Darmochwał and Yatsuka Nakamura. The topological space \mathcal{E}_T^2 . Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(5):617–621, 1991.
- [11] Agata Darmochwał and Yatsuka Nakamura. The topological space \mathcal{E}_T^2 . Simple closed curves. *Formalized Mathematics*, 2(5):663–664, 1991.
- [12] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(4):475–480, 1991.
- [13] Artur Kornilowicz, Robert Milewski, Adam Naumowicz, and Andrzej Trybulec. Gauges and cages. Part I. *Formalized Mathematics*, 9(3):501–509, 2001.
- [14] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(3):477–481, 1990.
- [15] Jarosław Kotowicz and Yatsuka Nakamura. Introduction to Go-board - part I. *Formalized Mathematics*, 3(1):107–115, 1992.
- [16] Robert Milewski. Upper and lower sequence of a cage. *Formalized Mathematics*, 9(4):787–790, 2001.
- [17] Yatsuka Nakamura and Czesław Byliński. Extremal properties of vertices on special polygons. Part I. *Formalized Mathematics*, 5(1):97–102, 1996.
- [18] Yatsuka Nakamura and Andrzej Trybulec. Decomposing a Go-board into cells. *Formalized Mathematics*, 5(3):323–328, 1996.
- [19] Yatsuka Nakamura and Andrzej Trybulec. A decomposition of a simple closed curves and the order of their points. *Formalized Mathematics*, 6(4):563–572, 1997.
- [20] Yatsuka Nakamura, Andrzej Trybulec, and Czesław Byliński. Bounded domains and unbounded domains. *Formalized Mathematics*, 8(1):1–13, 1999.
- [21] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(1):239–244, 1990.
- [22] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [23] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(4):777–780, 1990.
- [24] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [25] Andrzej Trybulec. On the decomposition of finite sequences. *Formalized Mathematics*, 5(3):317–322, 1996.
- [26] Andrzej Trybulec and Yatsuka Nakamura. On the order on a special polygon. *Formalized Mathematics*, 6(4):541–548, 1997.
- [27] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.

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