FORMALIZED MATHEMATICS
Volume 10, Number **3**, 2002
University of Białystok

On the Minimal Distance Between Sets in Euclidean Space[1]

Andrzej Trybulec
University of Białystok

Summary. The concept of the minimal distance between two sets in a Euclidean space is introduced and some useful lemmas are proved.

MML Identifier: `JORDAN1K`.

The papers [25], [26], [13], [24], [4], [27], [5], [1], [14], [17], [23], [8], [22], [15], [6], [3], [9], [10], [11], [2], [19], [21], [12], [20], [7], [16], and [18] provide the terminology and notation for this paper.

1. Preliminaries

In this paper X is a set and Y is a non empty set.

We now state several propositions:

(1) Let f be a function from X into Y. Suppose f is onto. Let y be an element of Y. Then there exists a set x such that $x \in X$ and $y = f(x)$.

(2) Let f be a function from X into Y. Suppose f is onto. Let y be an element of Y. Then there exists an element x of X such that $y = f(x)$.

(3) For every function f from X into Y and for every subset A of X such that f is onto holds $(f^\circ A)^{\rm c} \subseteq f^\circ A^{\rm c}$.

(4) For every function f from X into Y and for every subset A of X such that f is one-to-one holds $f^\circ A^{\rm c} \subseteq (f^\circ A)^{\rm c}$.

(5) For every function f from X into Y and for every subset A of X such that f is bijective holds $(f^\circ A)^{\rm c} = f^\circ A^{\rm c}$.

[1]This work has been partially supported by the European Community TYPES grant IST-1999-29001 and CALCULEMUS grant HPRN-CT-2000-00102. The work was completed while the author visited Shinhsu University (Nagano).

ISSN 1426–2630

2. Topological and Metrizable Spaces

One can prove the following two propositions:

(6) For every topological space T and for every subset A of T holds A is a component of $\emptyset_T$ iff A is empty.

(7) Let T be a non empty topological space and A, B, C be subsets of T. If $A \subseteq B$ and A is a component of C and B is a component of C, then $A = B$.

In the sequel n denotes a natural number.

We now state the proposition

(8) If $n \geqslant 1$, then for every subset P of $\mathcal{E}^n$ such that P is bounded holds $-P$ is not bounded.

In the sequel r is a real number and M is a non empty metric space.

Next we state a number of propositions:

(9) For every non empty subset C of $M_{\rm top}$ and for every point p of $M_{\rm top}$ holds $({\rm dist}_{\rm min}(C))(p) \geqslant 0$.

(10) Let C be a non empty subset of $M_{\rm top}$ and p be a point of M. If for every point q of M such that $q \in C$ holds $\rho(p, q) \geqslant r$, then $({\rm dist}_{\rm min}(C))(p) \geqslant r$.

(11) For all non empty subsets A, B of $M_{\rm top}$ holds ${\rm dist}^{\rm min}_{\rm min}(A, B) \geqslant 0$.

(12) For all compact subsets A, B of $M_{\rm top}$ such that A meets B holds ${\rm dist}^{\rm min}_{\rm min}(A, B) = 0$.

(13) Let A, B be non empty subsets of $M_{\rm top}$. Suppose that for all points p, q of M such that $p \in A$ and $q \in B$ holds $\rho(p, q) \geqslant r$. Then ${\rm dist}^{\rm min}_{\rm min}(A, B) \geqslant r$.

(14) Let P, Q be subsets of $\mathcal{E}^n_{\rm T}$. Suppose P is a component of $Q^{\rm c}$. Then P is inside component of Q or P is outside component of Q.

(15) If $n \geqslant 1$, then ${\rm BDD}\,\emptyset_{\mathcal{E}^n_{\rm T}} = \emptyset_{\mathcal{E}^n_{\rm T}}$.

(16) ${\rm BDD}\,\Omega_{\mathcal{E}^n_{\rm T}} = \emptyset_{\mathcal{E}^n_{\rm T}}$.

(17) If $n \geqslant 1$, then ${\rm UBD}\,\emptyset_{\mathcal{E}^n_{\rm T}} = \Omega_{\mathcal{E}^n_{\rm T}}$.

(18) ${\rm UBD}\,\Omega_{\mathcal{E}^n_{\rm T}} = \emptyset_{\mathcal{E}^n_{\rm T}}$.

(19) For every connected subset P of $\mathcal{E}^n_{\rm T}$ and for every subset Q of $\mathcal{E}^n_{\rm T}$ such that P misses Q holds $P \subseteq {\rm UBD}\,Q$ or $P \subseteq {\rm BDD}\,Q$.

3. Euclid Plane

For simplicity, we adopt the following rules: C, D are simple closed curves, n is a natural number, p, q, q_1, q_2 are points of $\mathcal{E}^2_{\rm T}$, r, s_1, s_2, t_1, t_2 are real numbers, and x, y are points of $\mathcal{E}^2$.

Next we state a number of propositions:

(20) $\rho([0, 0], r \cdot q) = |r| \cdot \rho([0, 0], q)$.

(21) $\rho(q_1 + q, q_2 + q) = \rho(q_1, q_2)$.
(22) If $p \neq q$, then $\rho(p, q) > 0$.
(23) $\rho(q_1 - q, q_2 - q) = \rho(q_1, q_2)$.
(24) $\rho(p, q) = \rho(-p, -q)$.
(25) $\rho(q - q_1, q - q_2) = \rho(q_1, q_2)$.
(26) $\rho(r \cdot p, r \cdot q) = |r| \cdot \rho(p, q)$.
(27) If $r \leqslant 1$, then $\rho(p, r \cdot p + (1 - r) \cdot q) = (1 - r) \cdot \rho(p, q)$.
(28) If $0 \leqslant r$, then $\rho(q, r \cdot p + (1 - r) \cdot q) = r \cdot \rho(p, q)$.
(29) If $p \in \mathcal{L}(q_1, q_2)$, then $\rho(q_1, p) + \rho(p, q_2) = \rho(q_1, q_2)$.
(30) If $q_1 \in \mathcal{L}(q_2, p)$ and $q_1 \neq q_2$, then $\rho(q_1, p) < \rho(q_2, p)$.
(31) If $y = [0, 0]$, then $\mathrm{Ball}(y, r) = \{q : |q| < r\}$.

4. Affine Maps

Next we state several propositions:

(32) $(\mathrm{AffineMap}(r, s_1, r, s_2))(p) = r \cdot p + [s_1, s_2]$.
(33) $(\mathrm{AffineMap}(r, \mathbf{q_1}, r, \mathbf{q_2}))(p) = r \cdot p + q$.
(34) If $s_1 > 0$ and $s_2 > 0$, then $\mathrm{AffineMap}(s_1, t_1, s_2, t_2) \cdot \mathrm{AffineMap}(\frac{1}{s_1}, -\frac{t_1}{s_1}, \frac{1}{s_2}, -\frac{t_2}{s_2}) = \mathrm{id}_{\mathcal{R}^2}$.
(35) If $y = [0, 0]$ and $x = q$ and $r > 0$, then $(\mathrm{AffineMap}(r, \mathbf{q_1}, r, \mathbf{q_2}))^\circ\, \mathrm{Ball}(y, 1) = \mathrm{Ball}(x, r)$.
(36) For all real numbers A, B, C, D such that $A > 0$ and $C > 0$ holds $\mathrm{AffineMap}(A, B, C, D)$ is onto.
(37) $\mathrm{Ball}(x, r)^{\mathrm{c}}$ is a connected subset of $\mathcal{E}^2_{\mathrm{T}}$.

5. Minimal Distance Between Subsets

Let us consider n and let A, B be subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\mathrm{dist}_{\min}(A, B)$ yielding a real number is defined by:

(Def. 1) There exist subsets A', B' of $(\mathcal{E}^n)_{\mathrm{top}}$ such that $A = A'$ and $B = B'$ and $\mathrm{dist}_{\min}(A, B) = \mathrm{dist}^{\min}_{\min}(A', B')$.

Let M be a non empty metric space and let P, Q be non empty compact subsets of M_{top}. Let us note that the functor $\mathrm{dist}^{\min}_{\min}(P, Q)$ is commutative. Let us observe that the functor $\mathrm{dist}^{\max}_{\max}(P, Q)$ is commutative.

Let us consider n and let A, B be non empty compact subsets of $\mathcal{E}^n_{\mathrm{T}}$. Let us observe that the functor $\mathrm{dist}_{\min}(A, B)$ is commutative.

Next we state several propositions:

(38) For all non empty subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{dist}_{\min}(A, B) \geqslant 0$.

(39) For all compact subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ such that A meets B holds $\mathrm{dist}_{\min}(A, B) = 0$.

(40) Let A, B be non empty subsets of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that for all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in A$ and $q \in B$ holds $\rho(p, q) \geqslant r$. Then $\mathrm{dist}_{\min}(A, B) \geqslant r$.

(41) Let D be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and A, C be non empty subsets of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. If $C \subseteq D$, then $\mathrm{dist}_{\min}(A, D) \leqslant \mathrm{dist}_{\min}(A, C)$.

(42) For all non empty compact subsets A, B of $\mathcal{E}^n_{\mathrm{T}}$ there exist points p, q of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in A$ and $q \in B$ and $\mathrm{dist}_{\min}(A, B) = \rho(p, q)$.

(43) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{dist}_{\min}(\{p\}, \{q\}) = \rho(p, q)$.

Let us consider n, let p be a point of $\mathcal{E}^n_{\mathrm{T}}$, and let B be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. The functor $\rho(p, B)$ yielding a real number is defined as follows:

(Def. 2) $\rho(p, B) = \mathrm{dist}_{\min}(\{p\}, B)$.

Next we state several propositions:

(44) For every non empty subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for every point p of $\mathcal{E}^n_{\mathrm{T}}$ holds $\rho(p, A) \geqslant 0$.

(45) For every compact subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for every point p of $\mathcal{E}^n_{\mathrm{T}}$ such that $p \in A$ holds $\rho(p, A) = 0$.

(46) Let A be a non empty compact subset of $\mathcal{E}^n_{\mathrm{T}}$ and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. Then there exists a point q of $\mathcal{E}^n_{\mathrm{T}}$ such that $q \in A$ and $\rho(p, A) = \rho(p, q)$.

(47) Let C be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$ and D be a subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$. If $C \subseteq D$, then for every point q of $\mathcal{E}^n_{\mathrm{T}}$ holds $\rho(q, D) \leqslant \rho(q, C)$.

(48) Let A be a non empty subset of $\mathcal{E}^n_{\mathrm{T}}$ and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. If for every point q of $\mathcal{E}^n_{\mathrm{T}}$ such that $q \in A$ holds $\rho(p, q) \geqslant r$, then $\rho(p, A) \geqslant r$.

(49) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ holds $\rho(p, \{q\}) = \rho(p, q)$.

(50) For every non empty subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ such that $q \in A$ holds $\rho(p, A) \leqslant \rho(p, q)$.

(51) Let A be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and B be an open subset of $\mathcal{E}^2_{\mathrm{T}}$. If $A \subseteq B$, then for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \notin B$ holds $\rho(p, B) < \rho(p, A)$.

6. BDD and UBD

The following two propositions are true:

(52) $\mathrm{UBD}\, C$ meets $\mathrm{UBD}\, D$.

(53) If $q \in \mathrm{UBD}\, C$ and $p \in \mathrm{BDD}\, C$, then $\rho(q, C) < \rho(q, p)$.

Let us consider C. Observe that $\mathrm{BDD}\, C$ is non empty.

One can prove the following three propositions:

(54) If $p \notin \mathrm{BDD}\, C$, then $\rho(p, C) \leqslant \rho(p, \mathrm{BDD}\, C)$.

(55) $C \not\subseteq \operatorname{BDD} D$ or $D \not\subseteq \operatorname{BDD} C$.

(56) If $C \subseteq \operatorname{BDD} D$, then $D \subseteq \operatorname{UBD} C$.

7. Main Definitions

We now state the proposition

(57) $\widetilde{\mathcal{L}}(\mathrm{Cage}(C, n)) \subseteq \operatorname{UBD} C$.

Let us consider C. The functor LowerMiddlePoint C yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 3) $\mathrm{LowerMiddlePoint}\, C = \mathrm{FPoint}(\mathrm{LowerArc}\, C, \text{W-min}\, C, \text{E-max}\, C, \mathrm{VerticalLine}\, \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2})$.

The functor UpperMiddlePoint C yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 4) $\mathrm{UpperMiddlePoint}\, C = \mathrm{FPoint}(\mathrm{UpperArc}\, C, \text{W-min}\, C, \text{E-max}\, C, \mathrm{VerticalLine}\, \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2})$.

We now state several propositions:

(58) $\mathrm{LowerArc}\, C$ meets $\mathrm{VerticalLine}\, \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(59) $\mathrm{UpperArc}\, C$ meets $\mathrm{VerticalLine}\, \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(60) $(\mathrm{LowerMiddlePoint}\, C)_{\mathbf{1}} = \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(61) $(\mathrm{UpperMiddlePoint}\, C)_{\mathbf{1}} = \frac{\text{W-bound}\, C + \text{E-bound}\, C}{2}$.

(62) $\mathrm{LowerMiddlePoint}\, C \in \mathrm{LowerArc}\, C$.

(63) $\mathrm{UpperMiddlePoint}\, C \in \mathrm{UpperArc}\, C$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Józef Białas and Yatsuka Nakamura. The theorem of Weierstrass. *Formalized Mathematics*, 5(**3**):353–359, 1996.

[3] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.

[7] Czesław Byliński and Mariusz Żynel. Cages - the external approximation of Jordan's curve. *Formalized Mathematics*, 9(**1**):19–24, 2001.

[8] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.

[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[10] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.

[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Simple closed curves. *Formalized Mathematics*, 2(**5**):663–664, 1991.

[12] Adam Grabowski and Yatsuka Nakamura. The ordering of points on a curve. Part II. *Formalized Mathematics*, 6(**4**):467–473, 1997.

[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[14] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.
[15] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[16] Artur Korniłowicz. Properties of left and right components. *Formalized Mathematics*, 8(**1**):163–168, 1999.
[17] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[18] Yatsuka Nakamura. On Outside Fashoda Meet Theorem. *Formalized Mathematics*, 9(**4**):697–704, 2001.
[19] Yatsuka Nakamura and Andrzej Trybulec. Decomposing a Go-board into cells. *Formalized Mathematics*, 5(**3**):323–328, 1996.
[20] Yatsuka Nakamura and Andrzej Trybulec. A decomposition of a simple closed curves and the order of their points. *Formalized Mathematics*, 6(**4**):563–572, 1997.
[21] Yatsuka Nakamura, Andrzej Trybulec, and Czesław Byliński. Bounded domains and unbounded domains. *Formalized Mathematics*, 8(**1**):1–13, 1999.
[22] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.
[23] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[24] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[25] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[26] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received August 19, 2002