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Free Order Sorted Universal Algebra[1]

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Summary. Free Order Sorted Universal Algebra — the general construction for any locally directed signatures.

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The papers [21], [13], [27], [32], [33], [11], [22], [12], [7], [10], [4], [18], [2], [20], [26], [14], [5], [3], [6], [1], [8], [25], [23], [17], [24], [9], [15], [16], [29], [31], [28], [30], and [19] provide the terminology and notation for this paper.

1. Preliminaries

In this paper S is an order sorted signature.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. A subset of U_0 is called an order sorted generator set of U_0 if:

(Def. 1) For every OSSubset O of U_0 such that $O = \mathrm{OSCl}\,\mathrm{it}$ holds the sorts of $\mathrm{OSGen}\,O$ = the sorts of U_0.

The following proposition is true

(1) Let S be an order sorted signature, U_0 be a strict non-empty order sorted algebra of S, and A be a subset of U_0. Then A is an order sorted generator set of U_0 if and only if for every OSSubset O of U_0 such that $O = \mathrm{OSCl}\,A$ holds $\mathrm{OSGen}\,O = U_0$.

Let us consider S, let U_0 be a monotone order sorted algebra of S, and let I_1 be an order sorted generator set of U_0. We say that I_1 is osfree if and only if the condition (Def. 2) is satisfied.

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(Def. 2) Let U_1 be a monotone non-empty order sorted algebra of S and f be a many sorted function from I_1 into the sorts of U_1. Then there exists a many sorted function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and order-sorted and $h \restriction I_1 = f$.

Let S be an order sorted signature and let I_1 be a monotone order sorted algebra of S. We say that I_1 is osfree if and only if:

(Def. 3) There exists an order sorted generator set of I_1 which is osfree.

2. Construction of Free Order Sorted Algebras for Given Signature

Let S be an order sorted signature and let X be a many sorted set indexed by S. The functor $\mathrm{OSREL}\,X$ yields a relation between $[\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!] \cup \bigcup \mathrm{coprod}(X)$ and $([\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!] \cup \bigcup \mathrm{coprod}(X))^*$ and is defined by the condition (Def. 4).

(Def. 4) Let a be an element of $[\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!] \cup \bigcup \mathrm{coprod}(X)$ and b be an element of $([\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!] \cup \bigcup \mathrm{coprod}(X))^*$. Then $\langle a, b \rangle \in \mathrm{OSREL}\,X$ if and only if the following conditions are satisfied:
 (i) $a \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!]$, and
 (ii) for every operation symbol o of S such that $\langle o$, the carrier of $S\rangle = a$ holds $\mathrm{len}\, b = \mathrm{len}\,\mathrm{Arity}(o)$ and for every set x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 \leqslant \mathrm{Arity}(o)_x$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then there exists an element i of the carrier of S such that $i \leqslant \mathrm{Arity}(o)_x$ and $b(x) \in \mathrm{coprod}(i, X)$.

In the sequel S is an order sorted signature, X is a many sorted set indexed by S, o is an operation symbol of S, and b is an element of $([\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!] \cup \bigcup \mathrm{coprod}(X))^*$.

One can prove the following proposition

(2) $\langle\langle o$, the carrier of $S\rangle, b\rangle \in \mathrm{OSREL}\,X$ if and only if the following conditions are satisfied:
 (i) $\mathrm{len}\, b = \mathrm{len}\,\mathrm{Arity}(o)$, and
 (ii) for every set x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$ the operation symbols of S, $\{$the carrier of $S\}\,:\!]$, then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 \leqslant \mathrm{Arity}(o)_x$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then there exists an element i of the carrier of S such that $i \leqslant \mathrm{Arity}(o)_x$ and $b(x) \in \mathrm{coprod}(i, X)$.

Let S be an order sorted signature and let X be a many sorted set indexed by S. The functor DTConOSA X yielding a tree construction structure is defined by:

(Def. 5) DTConOSA X = $\langle$[: the operation symbols of S, {the carrier of S} :] $\cup$ $\bigcup$coprod(X), OSREL $X\rangle$.

Let S be an order sorted signature and let X be a many sorted set indexed by S. Note that DTConOSA X is strict and non empty.

The following proposition is true

(3) Let S be an order sorted signature and X be a non-empty many sorted set indexed by S. Then the nonterminals of DTConOSA X = [: the operation symbols of S, {the carrier of S} :] and the terminals of DTConOSA $X = \bigcup$coprod(X).

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. Note that DTConOSA X has terminals, nonterminals, and useful nonterminals.

The following proposition is true

(4) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and t be a set. Then $t \in$ the terminals of DTConOSA X if and only if there exists an element s of the carrier of S and there exists a set x such that $x \in X(s)$ and $t = \langle x, s\rangle$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o be an operation symbol of S. The functor OSSym(o, X) yielding a symbol of DTConOSA X is defined as follows:

(Def. 6) OSSym(o, X) = $\langle o$, the carrier of $S\rangle$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. The functor ParsedTerms(X, s) yielding a subset of TS(DTConOSA X) is defined by the condition (Def. 7).

(Def. 7) ParsedTerms(X, s) = {a; a ranges over elements of TS(DTConOSA X): $\bigvee_{s_1\,:\,\text{element of the carrier of }S} \bigvee_{x\,:\,\text{set}} (s_1 \leqslant s \wedge x \in X(s_1) \wedge a =$ the root tree of $\langle x, s_1\rangle) \vee \bigvee_{o\,:\,\text{operation symbol of }S} (\langle o$, the carrier of $S\rangle = a(\emptyset) \wedge$ the result sort of $o \leqslant s)$}.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. Note that ParsedTerms(X, s) is non empty.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. The functor ParsedTerms X yields an order sorted set of S and is defined by:

(Def. 8) For every element s of the carrier of S holds (ParsedTerms X)(s) = ParsedTerms(X, s).

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. One can verify that ParsedTerms X is non-empty.

The following four propositions are true:

(5) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and x be a set. Suppose $x \in ((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$. Then x is a finite sequence of elements of TS(DTConOSA X).

(6) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and p be a finite sequence of elements of TS(DTConOSA X). Then $p \in ((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$ if and only if $\mathrm{dom}\,p = \mathrm{dom}\,\mathrm{Arity}(o)$ and for every natural number n such that $n \in \mathrm{dom}\,p$ holds $p(n) \in \mathrm{ParsedTerms}(X, \mathrm{Arity}(o)_n)$.

(7) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and p be a finite sequence of elements of TS(DTConOSA X). Then $\mathrm{OSSym}(o, X) \Rightarrow$ the roots of p if and only if $p \in ((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$.

(8) For every order sorted signature S and for every non-empty many sorted set X indexed by S holds $\bigcup \mathrm{rng}\,\mathrm{ParsedTerms}\,X = \mathrm{TS}(\mathrm{DTConOSA}\,X)$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o be an operation symbol of S. The functor $\mathrm{PTDenOp}(o, X)$ yields a function from $((\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S)(o)$ into $(\mathrm{ParsedTerms}\,X \cdot \text{the result sort of } S)(o)$ and is defined as follows:

(Def. 9) For every finite sequence p of elements of TS(DTConOSA X) such that $\mathrm{OSSym}(o, X) \Rightarrow$ the roots of p holds $(\mathrm{PTDenOp}(o, X))(p) = \mathrm{OSSym}(o, X)\text{-tree}(p)$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTOper X yields a many sorted function from $(\mathrm{ParsedTerms}\,X)^{\#} \cdot \text{the arity of } S$ into $\mathrm{ParsedTerms}\,X \cdot \text{the result sort of } S$ and is defined by:

(Def. 10) For every operation symbol o of S holds $(\mathrm{PTOper}\,X)(o) = \mathrm{PTDenOp}(o, X)$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. The functor ParsedTermsOSA X yielding an order sorted algebra of S is defined as follows:

(Def. 11) $\mathrm{ParsedTermsOSA}\,X = \langle \mathrm{ParsedTerms}\,X, \mathrm{PTOper}\,X \rangle$.

Let S be an order sorted signature and let X be a non-empty many sorted set indexed by S. One can check that ParsedTermsOSA X is strict and non-empty.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o be an operation symbol of S. Then $\mathrm{OSSym}(o, X)$ is a nonterminal of DTConOSA X.

Next we state several propositions:

(9) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and s be an element of the carrier of S. Then (the sorts of ParsedTermsOSA X)(s) $= \{a; a$ ranges over elements of TS(DTConOSA X): $\bigvee_{s_1 : \text{element of the carrier of } S} \bigvee_{x : \text{set}} (s_1 \leqslant s \ \wedge\ x \in X(s_1) \ \wedge\ a = \text{the root tree of } \langle x, s_1 \rangle) \ \vee\ \bigvee_{o : \text{operation symbol of } S} (\langle o, \text{the carrier of } S \rangle = a(\emptyset) \ \wedge\ \text{the result sort of } o \leqslant s)\}$.

(10) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, s, s_1 be elements of the carrier of S, and x be a set. Suppose $x \in X(s)$. Then

(i) the root tree of $\langle x, s \rangle$ is an element of TS(DTConOSA X),

(ii) for every set z holds $\langle z$, the carrier of $S \rangle \neq$ (the root tree of $\langle x, s \rangle)(\emptyset)$, and

(iii) the root tree of $\langle x, s \rangle \in$ (the sorts of ParsedTermsOSA X)(s_1) iff $s \leqslant s_1$.

(11) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, t be an element of TS(DTConOSA X), and o be an operation symbol of S. Suppose $t(\emptyset) = \langle o$, the carrier of $S \rangle$. Then

(i) there exists a subtree sequence p joinable by OSSym(o, X) such that $t =$ OSSym(o, X)-tree(p) and OSSym$(o, X) \Rightarrow$ the roots of p and $p \in$ Args$(o, \text{ParsedTermsOSA}\, X)$ and $t =$ (Den$(o, \text{ParsedTermsOSA}\, X))(p)$,

(ii) for every element s_2 of the carrier of S and for every set x holds $t \neq$ the root tree of $\langle x, s_2 \rangle$, and

(iii) for every element s_1 of the carrier of S holds $t \in$ (the sorts of ParsedTermsOSA X)(s_1) iff the result sort of $o \leqslant s_1$.

(12) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, n_1 be a symbol of DTConOSA X, and t_1 be a finite sequence of elements of TS(DTConOSA X). Suppose $n_1 \Rightarrow$ the roots of t_1. Then

(i) $n_1 \in$ the nonterminals of DTConOSA X,

(ii) n_1-tree$(t_1) \in$ TS(DTConOSA X), and

(iii) there exists an operation symbol o of S such that $n_1 = \langle o$, the carrier of $S \rangle$ and $t_1 \in$ Args$(o, \text{ParsedTermsOSA}\, X)$ and n_1-tree$(t_1) =$ (Den$(o$, ParsedTermsOSA X))(t_1) and for every element s_1 of the carrier of S holds n_1-tree$(t_1) \in$ (the sorts of ParsedTermsOSA X)(s_1) iff the result sort of $o \leqslant s_1$.

(13) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and x be a finite sequence. Then $x \in$ Args$(o, \text{ParsedTermsOSA}\, X)$ if and only if the following conditions are satisfied:

(i) x is a finite sequence of elements of TS(DTConOSA X), and

(ii) OSSym$(o, X) \Rightarrow$ the roots of x.

(14) Let S be an order sorted signature, X be a non-empty many sorted set

indexed by S, and t be an element of TS(DTConOSA X). Then there exists a sort symbol s of S such that $t \in$ (the sorts of ParsedTermsOSA X)(s) and for every element s_1 of the carrier of S such that $t \in$ (the sorts of ParsedTermsOSA X)(s_1) holds $s \leqslant s_1$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be an element of TS(DTConOSA X). The functor LeastSort t yields a sort symbol of S and is defined by the conditions (Def. 12).

(Def. 12)(i) $t \in$ (the sorts of ParsedTermsOSA X)(LeastSort t), and

(ii) for every element s_1 of the carrier of S such that $t \in$ (the sorts of ParsedTermsOSA X)(s_1) holds LeastSort $t \leqslant s_1$.

Let S be a non empty non void many sorted signature and let A be a non-empty algebra over S.

(Def. 13) An element of $\bigcup$(the sorts of A) is said to be an element of A.

We now state four propositions:

(15) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, and x be a set. Then x is an element of ParsedTermsOSA X if and only if x is an element of TS(DTConOSA X).

(16) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, and x be a set. If $x \in$ (the sorts of ParsedTermsOSA X)(s), then x is an element of TS(DTConOSA X).

(17) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, and x be a set. Suppose $x \in X(s)$. Let t be an element of TS(DTConOSA X). If $t =$ the root tree of $\langle x, s\rangle$, then LeastSort $t = s$.

(18) Let S be an order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, x be an element of Args(o, ParsedTermsOSA X), and t be an element of TS(DTConOSA X). If $t =$ (Den(o, ParsedTermsOSA X))(x), then LeastSort $t =$ the result sort of o.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let o_2 be an operation symbol of S. Note that Args(o_2, ParsedTermsOSA X) is non empty.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let x be a finite sequence of elements of TS(DTConOSA X). The functor LeastSorts x yielding an element of (the carrier of S)* is defined as follows:

(Def. 14) dom LeastSorts x = dom x and for every natural number y such that $y \in$ dom x there exists an element t of TS(DTConOSA X) such that $t = x(y)$ and (LeastSorts x)(y) = LeastSort t.

We now state the proposition

(19) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, o be an operation symbol of S, and x be a finite sequence of elements of TS(DTConOSA X). Then LeastSorts $x \leqslant$ Arity(o) if and only if $x \in$ Args(o, ParsedTermsOSA X).

Let us note that there exists a monotone order sorted signature which is locally directed and regular.

Let S be a locally directed regular monotone order sorted signature, let X be a non-empty many sorted set indexed by S, let o be an operation symbol of S, and let x be a finite sequence of elements of TS(DTConOSA X). Let us assume that OSSym(LBound(o, LeastSorts x), X) $\Rightarrow$ the roots of x. The functor $\pi_x o$ yields an element of TS(DTConOSA X) and is defined by:

(Def. 15) $\pi_x o =$ OSSym(LBound(o, LeastSorts x), X)-tree(x).

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be a symbol of DTConOSA X. Let us assume that there exists a finite sequence p such that $t \Rightarrow p$. The functor ${}^{@}(X, t)$ yields an operation symbol of S and is defined by:

(Def. 16) $\langle {}^{@}(X, t)$, the carrier of $S\rangle = t$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be a symbol of DTConOSA X. Let us assume that $t \in$ the terminals of DTConOSA X. The functor $\prod t$ yielding an element of TS(DTConOSA X) is defined by:

(Def. 17) $\prod t =$ the root tree of t.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor LCongruence X yielding a monotone order sorted congruence of ParsedTermsOSA X is defined by:

(Def. 18) For every monotone order sorted congruence R of ParsedTermsOSA X holds LCongruence $X \subseteq R$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor FreeOSA X yielding a strict non-empty monotone order sorted algebra of S is defined by:

(Def. 19) FreeOSA $X =$ QuotOSAlg(ParsedTermsOSA X, LCongruence X).

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, and let t be a symbol of DTConOSA X. The functor ${}^{@}t$ yields a subset of [: TS(DTConOSA X), the carrier of S :] and is defined by the condition (Def. 20).

(Def. 20) ${}^{@}t = \{\langle$the root tree of $t,\ s_1\rangle; s_1$ ranges over elements of the carrier of S: $\bigvee_{s\,:\,\text{element of the carrier of } S} \bigvee_{x\,:\,\text{set}} (x \in X(s) \ \wedge\ t = \langle x,\ s\rangle \ \wedge\ s \leqslant s_1)\}$.

Let S be an order sorted signature, let X be a non-empty many sorted set indexed by S, let n_1 be a symbol of DTConOSA X, and let x be a finite sequence of elements of $2^{[:\mathrm{TS}(\mathrm{DTConOSA}\,X),\,\text{the carrier of } S:]}$. The functor $^{@}(n_1, x)$ yielding a subset of $[:\mathrm{TS}(\mathrm{DTConOSA}\,X), \text{the carrier of } S:]$ is defined by the condition (Def. 21).

(Def. 21) $^{@}(n_1, x) = \{\langle(\mathrm{Den}(o_2, \mathrm{ParsedTermsOSA}\,X))(x_2), s_3\rangle; o_2$ ranges over operation symbols of S, x_2 ranges over elements of $\mathrm{Args}(o_2, \mathrm{ParsedTermsOSA}\,X)$, s_3 ranges over elements of the carrier of S: $\bigvee_{o_1 : \text{operation symbol of } S} (n_1 = \langle o_1, \text{the carrier of } S\rangle \wedge o_1 \cong o_2 \wedge \mathrm{len}\,\mathrm{Arity}(o_1) = \mathrm{len}\,\mathrm{Arity}(o_2) \wedge \text{the result sort of } o_1 \leqslant s_3 \wedge \text{the result sort of } o_2 \leqslant s_3) \wedge \bigvee_{w_3 : \text{element of (the carrier of } S)^*} (\mathrm{dom}\, w_3 = \mathrm{dom}\, x \wedge \bigwedge_{y : \text{natural number}} (y \in \mathrm{dom}\, x \Rightarrow \langle x_2(y), (w_3)_y\rangle \in x(y)))\}$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTClasses X yielding a function from $\mathrm{TS}(\mathrm{DTConOSA}\,X)$ into $2^{[:\mathrm{TS}(\mathrm{DTConOSA}\,X),\,\text{the carrier of } S:]}$ is defined by the conditions (Def. 22).

(Def. 22)(i) For every symbol t of DTConOSA X such that $t \in$ the terminals of DTConOSA X holds (PTClasses X)(the root tree of t) $= {}^{@}t$, and

(ii) for every symbol n_1 of DTConOSA X and for every finite sequence t_1 of elements of TS(DTConOSA X) and for every finite sequence r_1 such that $r_1 =$ the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of $2^{[:\mathrm{TS}(\mathrm{DTConOSA}\,X),\,\text{the carrier of } S:]}$ such that $x = \mathrm{PTClasses}\,X \cdot t_1$ holds (PTClasses X)(n_1-tree(t_1)) $= {}^{@}(n_1, x)$.

One can prove the following four propositions:

(20) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and t be an element of TS(DTConOSA X). Then

(i) for every element s of the carrier of S holds $t \in$ (the sorts of ParsedTermsOSA X)(s) iff $\langle t, s\rangle \in$ (PTClasses X)(t), and

(ii) for every element s of the carrier of S and for every element y of TS(DTConOSA X) such that $\langle y, s\rangle \in$ (PTClasses X)(t) holds $\langle t, s\rangle \in$ (PTClasses X)(y).

(21) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, t be an element of TS(DTConOSA X), and s be an element of the carrier of S. If there exists an element y of TS(DTConOSA X) such that $\langle y, s\rangle \in$ (PTClasses X)(t), then $\langle t, s\rangle \in$ (PTClasses X)(t).

(22) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, x, y be elements of TS(DTConOSA X), and s_1, s_2 be elements of the carrier of S. Suppose $s_1 \leqslant s_2$ and $x \in$ (the sorts of ParsedTermsOSA X)(s_1) and $y \in$ (the sorts of ParsedTermsOSA X)(s_1). Then $\langle y, s_1\rangle \in$ (PTClasses X)(x) if and only if $\langle y, s_2\rangle \in$ (PTClasses X)(x).

(23) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, x, y, z be elements of TS(DTConOSA X), and s be an element of the carrier of S. If $\langle y, s\rangle \in$ (PTClasses X)(x) and $\langle z, s\rangle \in$ (PTClasses X)(y), then $\langle x, s\rangle \in$ (PTClasses X)(z).

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTCongruence X yielding an equivalence order sorted relation of ParsedTermsOSA X is defined by the condition (Def. 23).

(Def. 23) Let i be a set. Suppose $i \in$ the carrier of S. Then (PTCongruence X)(i) = $\{\langle x, y\rangle$; x ranges over elements of TS(DTConOSA X), y ranges over elements of TS(DTConOSA X): $\langle x, i\rangle \in$ (PTClasses X)(y)$\}$.

One can prove the following propositions:

(24) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and x, y, s be sets. If $\langle x, s\rangle \in$ (PTClasses X)(y), then $x \in$ TS(DTConOSA X) and $y \in$ TS(DTConOSA X) and $s \in$ the carrier of S.

(25) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, C be a component of S, and x, y be sets. Then $\langle x, y\rangle \in$ CompClass(PTCongruence X, C) if and only if there exists an element s_1 of the carrier of S such that $s_1 \in C$ and $\langle x, s_1\rangle \in$ (PTClasses X)(y).

(26) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, and x be an element of (the sorts of ParsedTermsOSA X)(s). Then OSClass(PTCongruence X, x) = π_1((PTClasses X)(x)).

(27) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and R be a many sorted relation indexed by ParsedTermsOSA X. Then R = PTCongruence X if and only if the following conditions are satisfied:

(i) for all elements s_1, s_2 of the carrier of S and for every set x such that $x \in X(s_1)$ holds if $s_1 \leqslant s_2$, then $\langle$the root tree of $\langle x, s_1\rangle$, the root tree of $\langle x, s_1\rangle\rangle \in R(s_2)$ and for every set y such that $\langle$the root tree of $\langle x, s_1\rangle$, $y\rangle \in R(s_2)$ or $\langle y$, the root tree of $\langle x, s_1\rangle\rangle \in R(s_2)$ holds $s_1 \leqslant s_2$ and y = the root tree of $\langle x, s_1\rangle$, and

(ii) for all operation symbols o_1, o_2 of S and for every element x_1 of Args(o_1, ParsedTermsOSA X) and for every element x_2 of Args(o_2, ParsedTermsOSA X) and for every element s_3 of the carrier of S holds $\langle$(Den(o_1, ParsedTermsOSA X))(x_1), (Den(o_2, ParsedTermsOSA X))(x_2)$\rangle \in R(s_3)$ iff $o_1 \cong o_2$ and len Arity(o_1) = len Arity(o_2) and the result sort of $o_1 \leqslant s_3$ and the result sort of $o_2 \leqslant s_3$ and there exists an element w_3 of (the carrier of S)* such that dom w_3 = dom x_1 and for every natural number y such that $y \in$ dom w_3 holds $\langle x_1(y), x_2(y)\rangle \in R((w_3)_y)$.

(28) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then PTCongruence X is monotone.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. Observe that PTCongruence X is monotone.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. The functor PTVars(s, X) yields a subset of (the sorts of ParsedTermsOSA X)(s) and is defined by:

(Def. 24) For every set x holds $x \in$ PTVars(s, X) iff there exists a set a such that $a \in X(s)$ and $x =$ the root tree of $\langle a, s\rangle$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. One can check that PTVars(s, X) is non empty.

We now state the proposition

(29) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and s be an element of the carrier of S. Then PTVars$(s, X) = \{$the root tree of t; t ranges over symbols of DTConOSA X : $t \in$ the terminals of DTConOSA $X \wedge t_{\mathbf{2}} = s\}$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTVars X yielding a subset of ParsedTermsOSA X is defined by:

(Def. 25) For every element s of the carrier of S holds (PTVars X)(s) = PTVars(s, X).

The following proposition is true

(30) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then PTVars X is non-empty.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. The functor OSFreeGen(s, X) yields a subset of (the sorts of FreeOSA X)(s) and is defined by:

(Def. 26) For every set x holds $x \in$ OSFreeGen(s, X) iff there exists a set a such that $a \in X(s)$ and $x =$ (OSNatHom(ParsedTermsOSA X, LCongruence X)) (s)(the root tree of $\langle a, s\rangle$).

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. Note that OSFreeGen(s, X) is non empty.

We now state the proposition

(31) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, and s be an element of the carrier of S. Then $\mathrm{OSFreeGen}(s, X) = \{(\mathrm{OSNatHom}(\mathrm{ParsedTermsOSA}\, X, \mathrm{LCongruence}\, X))(s)(\text{the root tree of } t); t$ ranges over symbols of $\mathrm{DTConOSA}\, X : t \in$ the terminals of $\mathrm{DTConOSA}\, X \ \wedge\ t_{\mathbf{2}} = s\}$.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor $\mathrm{OSFreeGen}\, X$ yielding an order sorted generator set of $\mathrm{FreeOSA}\, X$ is defined by:

(Def. 27) For every element s of the carrier of S holds $(\mathrm{OSFreeGen}\, X)(s) = \mathrm{OSFreeGen}(s, X)$.

The following proposition is true

(32) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then $\mathrm{OSFreeGen}\, X$ is non-empty.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. Observe that $\mathrm{OSFreeGen}\, X$ is non-empty.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, let R be an order sorted congruence of $\mathrm{ParsedTermsOSA}\, X$, and let t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\, X)$. The functor $\mathrm{OSClass}(R, t)$ yielding an element of $\mathrm{OSClass}(R, \mathrm{LeastSort}\, t)$ is defined by the condition (Def. 28).

(Def. 28) Let s be an element of the carrier of S and x be an element of (the sorts of $\mathrm{ParsedTermsOSA}\, X)(s)$. If $t = x$, then $\mathrm{OSClass}(R, t) = \mathrm{OSClass}(R, x)$.

We now state several propositions:

(33) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, R be an order sorted congruence of $\mathrm{ParsedTermsOSA}\, X$, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\, X)$. Then $t \in \mathrm{OSClass}(R, t)$.

(34) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\, X)$, and x, x_1 be sets. Suppose $x \in X(s)$ and $t =$ the root tree of $\langle x, s \rangle$. Then $x_1 \in \mathrm{OSClass}(\mathrm{PTCongruence}\, X, t)$ if and only if $x_1 = t$.

(35) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, R be an order sorted congruence of $\mathrm{ParsedTermsOSA}\, X$, and t_2, t_3 be elements of $\mathrm{TS}(\mathrm{DTConOSA}\, X)$. Then $t_3 \in \mathrm{OSClass}(R, t_2)$ if and only if $\mathrm{OSClass}(R, t_2) = \mathrm{OSClass}(R, t_3)$.

(36) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, R_1, R_2 be order sorted congruences of $\mathrm{ParsedTermsOSA}\, X$, and t be an element of $\mathrm{TS}(\mathrm{DTConOSA}\, X)$. If $R_1 \subseteq R_2$, then $\mathrm{OSClass}(R_1, t) \subseteq \mathrm{OSClass}(R_2, t)$.

(37) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, s be an element of the carrier of S, t be an element of TS(DTConOSA X), and x, x_1 be sets. Suppose $x \in X(s)$ and $t =$ the root tree of $\langle x, s \rangle$. Then $x_1 \in$ OSClass(LCongruence X, t) if and only if $x_1 = t$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, let A be a non-empty many sorted set indexed by the carrier of S, let F be a many sorted function from PTVars X into A, and let t be a symbol of DTConOSA X. Let us assume that $t \in$ the terminals of DTConOSA X. The functor $\pi(F, A, t)$ yields an element of $\bigcup A$ and is defined as follows:

(Def. 29) For every function f such that $f = F(t_{\mathbf{2}})$ holds $\pi(F, A, t) = f$(the root tree of t).

Next we state the proposition

(38) Let S be a locally directed order sorted signature, X be a non-empty many sorted set indexed by S, U_1 be a monotone non-empty order sorted algebra of S, and f be a many sorted function from PTVars X into the sorts of U_1. Then there exists a many sorted function h from ParsedTermsOSA X into U_1 such that h is a homomorphism of ParsedTermsOSA X into U_1 and order-sorted and $h \restriction$ PTVars $X = f$.

Let S be a locally directed order sorted signature, let X be a non-empty many sorted set indexed by S, and let s be an element of the carrier of S. The functor NHReverse(s, X) yields a function from OSFreeGen(s, X) into PTVars(s, X) and is defined by the condition (Def. 30).

(Def. 30) Let t be a symbol of DTConOSA X.
Suppose (OSNatHom(ParsedTermsOSA X, LCongruence X))(s)(the root tree of t) $\in$ OSFreeGen(s, X). Then (NHReverse(s, X))((OSNatHom (ParsedTermsOSA X, LCongruence X))(s)(the root tree of t)) = the root tree of t.

Let S be a locally directed order sorted signature and let X be a non-empty many sorted set indexed by S. The functor NHReverse X yielding a many sorted function from OSFreeGen X into PTVars X is defined as follows:

(Def. 31) For every element s of the carrier of S holds (NHReverse X)(s) = NHReverse(s, X).

Next we state two propositions:

(39) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then OSFreeGen X is osfree.

(40) Let S be a locally directed order sorted signature and X be a non-empty many sorted set indexed by S. Then FreeOSA X is osfree.

Let S be a locally directed order sorted signature. Note that there exists a non-empty monotone order sorted algebra of S which is osfree and strict.

3. Minimal Terms

Let S be a locally directed regular monotone order sorted signature and let X be a non-empty many sorted set indexed by S. The functor PTMin X yields a function from TS(DTConOSA X) into TS(DTConOSA X) and is defined by the conditions (Def. 32).

(Def. 32)(i) For every symbol t of DTConOSA X such that $t \in$ the terminals of DTConOSA X holds (PTMin X)(the root tree of t) $= \prod t$, and

(ii) for every symbol n_1 of DTConOSA X and for every finite sequence t_1 of elements of TS(DTConOSA X) and for every finite sequence r_1 such that r_1 = the roots of t_1 and $n_1 \Rightarrow r_1$ and for every finite sequence x of elements of TS(DTConOSA X) such that $x = \mathrm{PTMin}\, X \cdot t_1$ holds $(\mathrm{PTMin}\, X)(n_1\text{-tree}(t_1)) = \pi_x({}^{@}(X, n_1))$.

Next we state several propositions:

(41) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t be an element of TS(DTConOSA X). Then

(i) $(\mathrm{PTMin}\, X)(t) \in \mathrm{OSClass}(\mathrm{PTCongruence}\, X, t)$,

(ii) $\mathrm{LeastSort}(\mathrm{PTMin}\, X)(t) \leqslant \mathrm{LeastSort}\, t$,

(iii) for every element s of the carrier of S and for every set x such that $x \in X(s)$ and t = the root tree of $\langle x, s\rangle$ holds $(\mathrm{PTMin}\, X)(t) = t$, and

(iv) for every operation symbol o of S and for every finite sequence t_1 of elements of TS(DTConOSA X) such that $\mathrm{OSSym}(o, X) \Rightarrow$ the roots of t_1 and $t = \mathrm{OSSym}(o, X)\text{-tree}(t_1)$ holds LeastSorts PTMin $X \cdot t_1 \leqslant \mathrm{Arity}(o)$ and $\mathrm{OSSym}(o, X) \Rightarrow$ the roots of PTMin $X \cdot t_1$ and $\mathrm{OSSym}(\mathrm{LBound}(o, \mathrm{LeastSorts}\,\mathrm{PTMin}\, X \cdot t_1), X) \Rightarrow$ the roots of PTMin $X \cdot t_1$ and $(\mathrm{PTMin}\, X)(t) = \mathrm{OSSym}(\mathrm{LBound}(o, \mathrm{LeastSorts}\,\mathrm{PTMin}\, X \cdot t_1), X)\text{-tree}(\mathrm{PTMin}\, X \cdot t_1)$.

(42) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t, t_2 be elements of TS(DTConOSA X). If $t_2 \in \mathrm{OSClass}(\mathrm{PTCongruence}\, X, t)$, then $(\mathrm{PTMin}\, X)(t_2) = (\mathrm{PTMin}\, X)(t)$.

(43) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t_2, t_3 be elements of TS(DTConOSA X). Then $t_3 \in \mathrm{OSClass}(\mathrm{PTCongruence}\, X, t_2)$ if and only if $(\mathrm{PTMin}\, X)(t_3) = (\mathrm{PTMin}\, X)(t_2)$.

(44) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and t_2 be

an element of TS(DTConOSA X). Then (PTMin X)((PTMin X)(t_2)) = (PTMin X)(t_2).

(45) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, R be a monotone equivalence order sorted relation of ParsedTermsOSA X, and t be an element of TS(DTConOSA X). Then $\langle t$, (PTMin X)(t)$\rangle \in R$(LeastSort t).

(46) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and R be a monotone equivalence order sorted relation of ParsedTermsOSA X. Then PTCongruence $X \subseteq R$.

(47) Let S be a locally directed regular monotone order sorted signature and X be a non-empty many sorted set indexed by S. Then LCongruence X = PTCongruence X.

Let S be a locally directed regular monotone order sorted signature and let X be a non-empty many sorted set indexed by S. An element of TS(DTConOSA X) is called a minimal term of S, X if:

(Def. 33) (PTMin X)(it) = it.

Let S be a locally directed regular monotone order sorted signature and let X be a non-empty many sorted set indexed by S. The functor MinTerms X yields a subset of TS(DTConOSA X) and is defined by:

(Def. 34) MinTerms X = rng PTMin X.

The following proposition is true

(48) Let S be a locally directed regular monotone order sorted signature, X be a non-empty many sorted set indexed by S, and x be a set. Then x is a minimal term of S, X if and only if $x \in$ MinTerms X.

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