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Order Sorted Algebras[1]

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Summary. Initial notions for order sorted algebras.

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The articles [9], [13], [14], [4], [15], [5], [8], [7], [2], [3], [1], [10], [12], [11], and [6] provide the notation and terminology for this paper.

1. Preliminaries

In this paper i is a set.

Let I be a set, let f be a many sorted set indexed by I, and let p be a finite sequence of elements of I. One can check that $f \cdot p$ is finite sequence-like.

Let S be a non empty many sorted signature. A sort symbol of S is an element of S.

Let S be a non empty many sorted signature.

(Def. 1) An element of the operation symbols of S is said to be an operation symbol of S.

Let S be a non void non empty many sorted signature and let o be an operation symbol of S. Then the result sort of o is an element of S.

Let X be a set. Then $\triangle_X$ is an order in X. We introduce $\triangle^o_X$ as a synonym of $\triangle_X$.

Let X be a set. Then $\triangle_X$ is an equivalence relation of X. We introduce $\triangle^r_X$ as a synonym of $\triangle_X$.

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We introduce overloaded many sorted signatures which are extensions of many sorted signature and are systems

$\langle$ a carrier, operation symbols, an overloading, an arity, a result sort $\rangle$,
where the carrier is a set, the operation symbols constitute a set, the overloading is an equivalence relation of the operation symbols, the arity is a function from the operation symbols into the carrier*, and the result sort is a function from the operation symbols into the carrier.

We introduce relation sorted signatures which are extensions of many sorted signature and relational structure and are systems

$\langle$ a carrier, an internal relation, operation symbols, an arity, a result sort $\rangle$,
where the carrier is a set, the internal relation is a binary relation on the carrier, the operation symbols constitute a set, the arity is a function from the operation symbols into the carrier*, and the result sort is a function from the operation symbols into the carrier.

We consider overloaded relation sorted signatures as extensions of overloaded many sorted signature and relation sorted signature as systems

$\langle$ a carrier, an internal relation, operation symbols, an overloading, an arity, a result sort $\rangle$,
where the carrier is a set, the internal relation is a binary relation on the carrier, the operation symbols constitute a set, the overloading is an equivalence relation of the operation symbols, the arity is a function from the operation symbols into the carrier*, and the result sort is a function from the operation symbols into the carrier.

For simplicity, we use the following convention: A, O are non empty sets, R is an order in A, O_1 is an equivalence relation of O, f is a function from O into A^*, and g is a function from O into A.

One can prove the following proposition

(1) $\langle A, R, O, O_1, f, g\rangle$ is non empty, non void, reflexive, transitive, and antisymmetric.

Let us consider A, R, O, O_1, f, g. One can verify that $\langle A, R, O, O_1, f, g\rangle$ is strict, non empty, reflexive, transitive, and antisymmetric.

2. The Notions: Order-Sorted, Discernable, Op-Discrete

In the sequel S is an overloaded relation sorted signature.

Let us consider S. We say that S is order-sorted if and only if:

(Def. 2) S is reflexive, transitive, and antisymmetric.

Let us note that every overloaded relation sorted signature which is order-sorted is also reflexive, transitive, and antisymmetric and there exists an overloaded relation sorted signature which is strict, non empty, non void, and order-sorted.

Let us observe that there exists an overloaded many sorted signature which is non empty and non void.

Let S be a non empty non void overloaded many sorted signature and let x, y be operation symbols of S. The predicate $x \cong y$ is defined by:

(Def. 3) $\langle x, y \rangle \in$ the overloading of S.

Let us notice that the predicate $x \cong y$ is reflexive and symmetric.

One can prove the following proposition

(2) Let S be a non empty non void overloaded many sorted signature and o, o_1, o_2 be operation symbols of S. If $o \cong o_1$ and $o_1 \cong o_2$, then $o \cong o_2$.

Let S be a non empty non void overloaded many sorted signature. We say that S is discernable if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let x, y be operation symbols of S. Suppose $x \cong y$ and $\mathrm{Arity}(x) = \mathrm{Arity}(y)$ and the result sort of $x =$ the result sort of y. Then $x = y$.

We say that S is op-discrete if and only if:

(Def. 5) The overloading of $S = \triangle^r_{\text{the operation symbols of } S}$.

The following two propositions are true:

(3) Let S be a non empty non void overloaded many sorted signature. Then S is op-discrete if and only if for all operation symbols x, y of S such that $x \cong y$ holds $x = y$.

(4) For every non empty non void overloaded many sorted signature S such that S is op-discrete holds S is discernable.

3. Order Sorted Signature

In the sequel S_0 is a non empty non void many sorted signature.

Let us consider S_0. The functor $\mathrm{OSSign}\, S_0$ yields a strict non empty non void order-sorted overloaded relation sorted signature and is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of $S_0 =$ the carrier of $\mathrm{OSSign}\, S_0$,

(ii) $\triangle_{\text{the carrier of } S_0} =$ the internal relation of $\mathrm{OSSign}\, S_0$,

(iii) the operation symbols of $S_0 =$ the operation symbols of $\mathrm{OSSign}\, S_0$,

(iv) $\triangle_{\text{the operation symbols of } S_0} =$ the overloading of $\mathrm{OSSign}\, S_0$,

(v) the arity of $S_0 =$ the arity of $\mathrm{OSSign}\, S_0$, and

(vi) the result sort of $S_0 =$ the result sort of $\mathrm{OSSign}\, S_0$.

Next we state the proposition

(5) $\mathrm{OSSign}\, S_0$ is discrete and op-discrete.

Let us mention that there exists a strict non empty non void order-sorted overloaded relation sorted signature which is discrete, op-discrete, and discernable.

Let us observe that every non empty non void overloaded relation sorted signature which is op-discrete is also discernable.

Let us consider S_0. Observe that OSSign S_0 is discrete and op-discrete.

An order sorted signature is a discernable non empty non void order-sorted overloaded relation sorted signature.

We use the following convention: S is a non empty poset, s_1, s_2 are elements of S, and w_1, w_2 are elements of (the carrier of S)*.

Let us consider S and let w_1, w_2 be elements of (the carrier of S)*. The predicate $w_1 \leqslant w_2$ is defined as follows:

(Def. 7) $\operatorname{len} w_1 = \operatorname{len} w_2$ and for every set i such that $i \in \operatorname{dom} w_1$ and for all s_1, s_2 such that $s_1 = w_1(i)$ and $s_2 = w_2(i)$ holds $s_1 \leqslant s_2$.

Let us note that the predicate $w_1 \leqslant w_2$ is reflexive.

We now state two propositions:

(6) For all elements w_1, w_2 of (the carrier of S)* such that $w_1 \leqslant w_2$ and $w_2 \leqslant w_1$ holds $w_1 = w_2$.

(7) If S is discrete and $w_1 \leqslant w_2$, then $w_1 = w_2$.

We follow the rules: S is an order sorted signature, o, o_1, o_2 are operation symbols of S, and w_1 is an element of (the carrier of S)*.

One can prove the following proposition

(8) If S is discrete and $o_1 \cong o_2$ and $\operatorname{Arity}(o_1) \leqslant \operatorname{Arity}(o_2)$ and the result sort of $o_1 \leqslant$ the result sort of o_2, then $o_1 = o_2$.

Let us consider S and let us consider o. We say that o is monotone if and only if:

(Def. 8) For every o_2 such that $o \cong o_2$ and $\operatorname{Arity}(o) \leqslant \operatorname{Arity}(o_2)$ holds the result sort of $o \leqslant$ the result sort of o_2.

Let us consider S. We say that S is monotone if and only if:

(Def. 9) Every operation symbol of S is monotone.

The following proposition is true

(9) If S is op-discrete, then S is monotone.

Let us observe that there exists an order sorted signature which is monotone.

Let S be a monotone order sorted signature. Observe that there exists an operation symbol of S which is monotone.

Let S be a monotone order sorted signature. One can check that every operation symbol of S is monotone.

One can check that every order sorted signature which is op-discrete is also monotone.

We now state the proposition

(10) If S is monotone and $\operatorname{Arity}(o_1) = \emptyset$ and $o_1 \cong o_2$ and $\operatorname{Arity}(o_2) = \emptyset$, then $o_1 = o_2$.

Let us consider S, o, o_1, w_1. We say that o_1 has least args for o, w_1 if and only if:

(Def. 10) $o \cong o_1$ and $w_1 \leqslant \mathrm{Arity}(o_1)$ and for every o_2 such that $o \cong o_2$ and $w_1 \leqslant \mathrm{Arity}(o_2)$ holds $\mathrm{Arity}(o_1) \leqslant \mathrm{Arity}(o_2)$.

We say that o_1 has least sort for o, w_1 if and only if:

(Def. 11) $o \cong o_1$ and $w_1 \leqslant \mathrm{Arity}(o_1)$ and for every o_2 such that $o \cong o_2$ and $w_1 \leqslant \mathrm{Arity}(o_2)$ holds the result sort of $o_1 \leqslant$ the result sort of o_2.

Let us consider S, o, o_1, w_1. We say that o_1 has least rank for o, w_1 if and only if:

(Def. 12) o_1 has least args for o, w_1 and least sort for o, w_1.

Let us consider S, o. We say that o is regular if and only if:

(Def. 13) o is monotone and for every w_1 such that $w_1 \leqslant \mathrm{Arity}(o)$ holds there exists o_1 which has least args for o, w_1.

Let S_1 be a monotone order sorted signature. We say that S_1 is regular if and only if:

(Def. 14) Every operation symbol of S_1 is regular.

In the sequel S_1 is a monotone order sorted signature, o, o_1 are operation symbols of S_1, and w_1 is an element of (the carrier of S_1)*.

We now state two propositions:

(11) S_1 is regular iff for all o, w_1 such that $w_1 \leqslant \mathrm{Arity}(o)$ holds there exists o_1 which has least rank for o, w_1.

(12) For every monotone order sorted signature S_1 such that S_1 is op-discrete holds S_1 is regular.

One can verify that there exists a monotone order sorted signature which is regular.

Let us mention that every monotone order sorted signature which is op-discrete is also regular.

Let S_2 be a regular monotone order sorted signature. One can verify that every operation symbol of S_2 is regular.

We adopt the following rules: S_2 is a regular monotone order sorted signature, o, o_3, o_4 are operation symbols of S_2, and w_1 is an element of (the carrier of S_2)*.

One can prove the following proposition

(13) If $w_1 \leqslant \mathrm{Arity}(o)$ and o_3 has least args for o, w_1 and o_4 has least args for o, w_1, then $o_3 = o_4$.

Let us consider S_2, o, w_1. Let us assume that $w_1 \leqslant \mathrm{Arity}(o)$. The functor $\mathrm{LBound}(o, w_1)$ yields an operation symbol of S_2 and is defined as follows:

(Def. 15) $\mathrm{LBound}(o, w_1)$ has least args for o, w_1.

One can prove the following proposition

(14) For every w_1 such that $w_1 \leqslant \mathrm{Arity}(o)$ holds $\mathrm{LBound}(o, w_1)$ has least rank for o, w_1.

In the sequel R denotes a non empty poset and z denotes a non empty set.

Let us consider R, z. The functor $\mathrm{ConstOSSet}(R, z)$ yielding a many sorted set indexed by the carrier of R is defined by:

(Def. 16) $\mathrm{ConstOSSet}(R, z) = (\text{the carrier of } R) \longmapsto z$.

The following proposition is true

(15) $\mathrm{ConstOSSet}(R, z)$ is non-empty and for all elements s_1, s_2 of R such that $s_1 \leqslant s_2$ holds $(\mathrm{ConstOSSet}(R, z))(s_1) \subseteq (\mathrm{ConstOSSet}(R, z))(s_2)$.

Let C be a 1-sorted structure.

(Def. 17) A many sorted set indexed by the carrier of C is said to be a many sorted set indexed by C.

Let us consider R, z. Then $\mathrm{ConstOSSet}(R, z)$ is a many sorted set indexed by R.

Let us consider R and let M be a many sorted set indexed by R. We say that M is order-sorted if and only if:

(Def. 18) For all elements s_1, s_2 of R such that $s_1 \leqslant s_2$ holds $M(s_1) \subseteq M(s_2)$.

Next we state the proposition

(16) $\mathrm{ConstOSSet}(R, z)$ is order-sorted.

Let us consider R. Observe that there exists a many sorted set indexed by R which is order-sorted.

Let us consider R, z. Then $\mathrm{ConstOSSet}(R, z)$ is an order-sorted many sorted set indexed by R.

Let R be a non empty poset. An order sorted set of R is an order-sorted many sorted set indexed by R.

Let R be a non empty poset. Observe that there exists an order sorted set of R which is non-empty.

We adopt the following convention: s_1, s_2 denote sort symbols of S, o, o_1, o_2, o_3 denote operation symbols of S, and w_1, w_2 denote elements of $(\text{the carrier of } S)^*$.

Let us consider S and let M be an algebra over S. We say that M is order-sorted if and only if:

(Def. 19) For all s_1, s_2 such that $s_1 \leqslant s_2$ holds $(\text{the sorts of } M)(s_1) \subseteq (\text{the sorts of } M)(s_2)$.

The following proposition is true

(17) For every algebra M over S holds M is order-sorted iff the sorts of M are an order sorted set of S.

In the sequel C_1 denotes a many sorted function from $(\mathrm{ConstOSSet}(S, z))^{\#} \cdot$ the arity of S into $\mathrm{ConstOSSet}(S, z) \cdot$ the result sort of S.

Let us consider S, z, C_1. The functor $\text{ConstOSA}(S, z, C_1)$ yielding a strict non-empty algebra over S is defined by:

(Def. 20) The sorts of $\text{ConstOSA}(S, z, C_1) = \text{ConstOSSet}(S, z)$ and the characteristics of $\text{ConstOSA}(S, z, C_1) = C_1$.

One can prove the following proposition

(18) $\text{ConstOSA}(S, z, C_1)$ is order-sorted.

Let us consider S. One can check that there exists an algebra over S which is strict, non-empty, and order-sorted.

Let us consider S, z, C_1. One can verify that $\text{ConstOSA}(S, z, C_1)$ is order-sorted.

Let us consider S. An order sorted algebra of S is an order-sorted algebra over S.

Next we state the proposition

(19) For every discrete order sorted signature S holds every algebra over S is order-sorted.

Let S be a discrete order sorted signature. Observe that every algebra over S is order-sorted.

In the sequel A denotes an order sorted algebra of S.

We now state the proposition

(20) If $w_1 \leqslant w_2$, then (the sorts of $A)^{\#}(w_1) \subseteq$ (the sorts of $A)^{\#}(w_2)$.

In the sequel M is an algebra over S_0.

Let us consider S_0, M. The functor $\text{OSAlg}\, M$ yielding a strict order sorted algebra of $\text{OSSign}\, S_0$ is defined as follows:

(Def. 21) The sorts of $\text{OSAlg}\, M$ = the sorts of M and the characteristics of $\text{OSAlg}\, M$ = the characteristics of M.

In the sequel A denotes an order sorted algebra of S.

We now state the proposition

(21) For all elements w_1, w_2, w_3 of (the carrier of $S)^*$ such that $w_1 \leqslant w_2$ and $w_2 \leqslant w_3$ holds $w_1 \leqslant w_3$.

Let us consider S, o_1, o_2. The predicate $o_1 \leqslant o_2$ is defined as follows:

(Def. 22) $o_1 \cong o_2$ and $\text{Arity}(o_1) \leqslant \text{Arity}(o_2)$ and the result sort of $o_1 \leqslant$ the result sort of o_2.

Let us note that the predicate $o_1 \leqslant o_2$ is reflexive.

We now state several propositions:

(22) If $o_1 \leqslant o_2$ and $o_2 \leqslant o_1$, then $o_1 = o_2$.

(23) If $o_1 \leqslant o_2$ and $o_2 \leqslant o_3$, then $o_1 \leqslant o_3$.

(24) If the result sort of $o_1 \leqslant$ the result sort of o_2, then $\text{Result}(o_1, A) \subseteq \text{Result}(o_2, A)$.

(25) If $\text{Arity}(o_1) \leqslant \text{Arity}(o_2)$, then $\text{Args}(o_1, A) \subseteq \text{Args}(o_2, A)$.

(26) If $o_1 \leqslant o_2$, then $\mathrm{Args}(o_1, A) \subseteq \mathrm{Args}(o_2, A)$ and $\mathrm{Result}(o_1, A) \subseteq \mathrm{Result}(o_2, A)$.

Let us consider S, A. We say that A is monotone if and only if:

(Def. 23) For all o_1, o_2 such that $o_1 \leqslant o_2$ holds $\mathrm{Den}(o_2, A) \restriction \mathrm{Args}(o_1, A) = \mathrm{Den}(o_1, A)$.

We now state two propositions:

(27) Let A be a non-empty order sorted algebra of S. Then A is monotone if and only if for all o_1, o_2 such that $o_1 \leqslant o_2$ holds $\mathrm{Den}(o_1, A) \subseteq \mathrm{Den}(o_2, A)$.

(28) If S is discrete and op-discrete, then A is monotone.

Let us consider S, z and let z_1 be an element of z. The functor $\mathrm{TrivialOSA}(S, z, z_1)$ yielding a strict order sorted algebra of S is defined by:

(Def. 24) The sorts of $\mathrm{TrivialOSA}(S, z, z_1) = \mathrm{ConstOSSet}(S, z)$ and for every o holds $\mathrm{Den}(o, \mathrm{TrivialOSA}(S, z, z_1)) = \mathrm{Args}(o, \mathrm{TrivialOSA}(S, z, z_1)) \longmapsto z_1$.

Next we state the proposition

(29) For every element z_1 of z holds $\mathrm{TrivialOSA}(S, z, z_1)$ is non-empty and $\mathrm{TrivialOSA}(S, z, z_1)$ is monotone.

Let us consider S. Note that there exists an order sorted algebra of S which is monotone, strict, and non-empty.

Let us consider S, z and let z_1 be an element of z. One can check that $\mathrm{TrivialOSA}(S, z, z_1)$ is monotone and non-empty.

In the sequel o_5, o_6 are operation symbols of S.

Let us consider S. The functor $\mathrm{OperNames}\, S$ yields a non empty family of subsets of the operation symbols of S and is defined as follows:

(Def. 25) $\mathrm{OperNames}\, S = \mathrm{Classes}\,(\text{the overloading of } S)$.

Let us consider S. One can check that every element of $\mathrm{OperNames}\, S$ is non empty.

Let us consider S. An OperName of S is an element of $\mathrm{OperNames}\, S$.

Let us consider S, o_5. The functor $\mathrm{Name}\, o_5$ yields an OperName of S and is defined by:

(Def. 26) $\mathrm{Name}\, o_5 = [o_5]_{\text{the overloading of } S}$.

Next we state three propositions:

(30) $o_5 \cong o_6$ iff $o_6 \in [o_5]_{\text{the overloading of } S}$.

(31) $o_5 \cong o_6$ iff $\mathrm{Name}\, o_5 = \mathrm{Name}\, o_6$.

(32) For every set X holds X is an OperName of S iff there exists o_5 such that $X = \mathrm{Name}\, o_5$.

Let us consider S and let o be an OperName of S. We see that the element of o is an operation symbol of S.

Next we state two propositions:

(33) For every OperName o_8 of S and for every operation symbol o_7 of S holds o_7 is an element of o_8 iff $\mathrm{Name}\, o_7 = o_8$.

(34) Let S_2 be a regular monotone order sorted signature, o_5, o_6 be operation symbols of S_2, and w be an element of (the carrier of S_2)*. If $o_5 \cong o_6$ and $\mathrm{len}\,\mathrm{Arity}(o_5) = \mathrm{len}\,\mathrm{Arity}(o_6)$ and $w \leqslant \mathrm{Arity}(o_5)$ and $w \leqslant \mathrm{Arity}(o_6)$, then $\mathrm{LBound}(o_5, w) = \mathrm{LBound}(o_6, w)$.

Let S_2 be a regular monotone order sorted signature, let o_8 be an OperName of S_2, and let w be an element of (the carrier of S_2)*. Let us assume that there exists an element o_7 of o_8 such that $w \leqslant \mathrm{Arity}(o_7)$. The functor $\mathrm{LBound}(o_8, w)$ yields an element of o_8 and is defined as follows:

(Def. 27) For every element o_7 of o_8 such that $w \leqslant \mathrm{Arity}(o_7)$ holds $\mathrm{LBound}(o_8, w) = \mathrm{LBound}(o_7, w)$.

Next we state the proposition

(35) Let S be a regular monotone order sorted signature, o be an operation symbol of S, and w_1 be an element of (the carrier of S)*. If $w_1 \leqslant \mathrm{Arity}(o)$, then $\mathrm{LBound}(o, w_1) \leqslant o$.

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