FORMALIZED MATHEMATICS
Volume 10, Number **3**, 2002
University of Białystok

Subalgebras of an Order Sorted Algebra. Lattice of Subalgebras[1]

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MML Identifier: OSALG_2.

The papers [8], [5], [12], [14], [4], [7], [15], [3], [1], [6], [9], [10], [11], [2], and [13] provide the notation and terminology for this paper.

1. Auxiliary Facts about Order Sorted Sets

In this paper x denotes a set and R denotes a non empty poset.

Next we state two propositions:

(1) For all order sorted sets X, Y of R holds $X \cap Y$ is an order sorted set of R.

(2) For all order sorted sets X, Y of R holds $X \cup Y$ is an order sorted set of R.

Let R be a non empty poset and let M be an order sorted set of R. A many sorted subset indexed by M is said to be an order sorted subset of M if:

(Def. 1) It is an order sorted set of R.

Let R be a non empty poset and let M be a non-empty order sorted set of R. Note that there exists an order sorted subset of M which is non-empty.

[1]This work was done during author's research visit in Bialystok, funded by the CALCULEMUS grant HPRN-CT-2000-00102.

ISSN 1426–2630

2. Constants of an Order Sorted Algebra

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. A many sorted subset indexed by the sorts of U_0 is said to be an OSSubset of U_0 if:

(Def. 2) It is an order sorted set of S.

Let S be an order sorted signature. Note that there exists an order sorted algebra of S which is monotone, strict, and non-empty.

Let S be an order sorted signature and let U_0 be a non-empty order sorted algebra of S. One can verify that there exists an OSSubset of U_0 which is non-empty.

Next we state the proposition

(3) For every non void strict non empty many sorted signature S_0 with constant operations holds OSSign S_0 has constant operations.

Let us note that there exists an order sorted signature which is strict and has constant operations.

3. Subalgebras of an Order Sorted Algebra

The following proposition is true

(4) Let S be an order sorted signature and U_0 be an order sorted algebra of S. Then $\langle$the sorts of U_0, the characteristics of $U_0\rangle$ is order-sorted.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. One can verify that there exists a subalgebra of U_0 which is order-sorted.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. An OSSubAlgebra of U_0 is an order-sorted subalgebra of U_0.

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. One can verify that there exists an OSSubAlgebra of U_0 which is strict.

Let S be an order sorted signature and let U_0 be a non-empty order sorted algebra of S. Observe that there exists an OSSubAlgebra of U_0 which is non-empty and strict.

One can prove the following proposition

(5) Let S be an order sorted signature, U_0 be an order sorted algebra of S, and U_1 be an algebra over S. Then U_1 is an OSSubAlgebra of U_0 if and only if the following conditions are satisfied:

(i) the sorts of U_1 are an OSSubset of U_0, and

(ii) for every OSSubset B of U_0 such that $B =$ the sorts of U_1 holds B is operations closed and the characteristics of $U_1 = \mathrm{Opers}(U_0, B)$.

In the sequel S_1 is an order sorted signature and O_0, O_1, O_2 are order sorted algebras of S_1.

The following propositions are true:

(6) O_1 is an OSSubAlgebra of O_1.

(7) If O_0 is an OSSubAlgebra of O_1 and O_1 is an OSSubAlgebra of O_2, then O_0 is an OSSubAlgebra of O_2.

(8) If O_1 is a strict OSSubAlgebra of O_2 and O_2 is a strict OSSubAlgebra of O_1, then $O_1 = O_2$.

(9) For all OSSubAlgebras O_1, O_2 of O_0 such that the sorts of $O_1 \subseteq$ the sorts of O_2 holds O_1 is an OSSubAlgebra of O_2.

(10) For all strict OSSubAlgebras O_1, O_2 of O_0 such that the sorts of $O_1 =$ the sorts of O_2 holds $O_1 = O_2$.

In the sequel s, s_1, s_2 are sort symbols of S_1.

Let us consider S_1, O_0, s. The functor $\mathrm{OSConstants}(O_0, s)$ yields a subset of (the sorts of O_0)(s) and is defined by:

(Def. 3) $\mathrm{OSConstants}(O_0, s) = \bigcup\{\mathrm{Constants}(O_0, s_2) : s_2 \leqslant s\}$.

One can prove the following proposition

(11) $\mathrm{Constants}(O_0, s) \subseteq \mathrm{OSConstants}(O_0, s)$.

Let us consider S_1 and let M be a many sorted set indexed by the carrier of S_1. The functor $\mathrm{OSCl}\, M$ yields an order sorted set of S_1 and is defined by:

(Def. 4) For every sort symbol s of S_1 holds $(\mathrm{OSCl}\, M)(s) = \bigcup\{M(s_1) : s_1 \leqslant s\}$.

Next we state three propositions:

(12) For every many sorted set M indexed by the carrier of S_1 holds $M \subseteq \mathrm{OSCl}\, M$.

(13) Let M be a many sorted set indexed by the carrier of S_1 and A be an order sorted set of S_1. If $M \subseteq A$, then $\mathrm{OSCl}\, M \subseteq A$.

(14) For every order sorted signature S and for every order sorted set X of S holds $\mathrm{OSCl}\, X = X$.

Let us consider S_1, O_0. The functor $\mathrm{OSConstants}\, O_0$ yields an OSSubset of O_0 and is defined by:

(Def. 5) For every sort symbol s of S_1 holds $(\mathrm{OSConstants}\, O_0)(s) = \mathrm{OSConstants}(O_0, s)$.

One can prove the following propositions:

(15) $\mathrm{Constants}(O_0) \subseteq \mathrm{OSConstants}\, O_0$.

(16) For every OSSubset A of O_0 such that $\mathrm{Constants}(O_0) \subseteq A$ holds $\mathrm{OSConstants}\, O_0 \subseteq A$.

(17) For every OSSubset A of O_0 holds $\mathrm{OSConstants}\, O_0 = \mathrm{OSCl}\,\mathrm{Constants}(O_0)$.

(18) For every OSSubAlgebra O_1 of O_0 holds $\mathrm{OSConstants}\, O_0$ is an OSSubset of O_1.

(19) Let S be an order sorted signature with constant operations, O_0 be a non-empty order sorted algebra of S, and O_1 be a non-empty OSSubAlgebra of O_0. Then $\mathrm{OSConstants}\, O_0$ is a non-empty OSSubset of O_1.

4. Order Sorted Subsets of an Order Sorted Algebra

Next we state the proposition

(20) Let I be a set, M be a many sorted set indexed by I, and x be a set. Then x is a many sorted subset indexed by M if and only if $x \in \prod(2^M)$.

Let R be a non empty poset and let M be an order sorted set of R. The functor OSbool M yielding a set is defined by:

(Def. 6) For every set x holds $x \in$ OSbool M iff x is an order sorted subset of M.

Let S be an order sorted signature, let U_0 be an order sorted algebra of S, and let A be an OSSubset of U_0. The functor OSSubSort A yields a set and is defined as follows:

(Def. 7) OSSubSort $A = \{x; x$ ranges over elements of SubSorts(A): x is an order sorted set of $S\}$.

We now state two propositions:

(21) For every OSSubset A of O_0 holds OSSubSort $A \subseteq$ SubSorts(A).

(22) For every OSSubset A of O_0 holds the sorts of $O_0 \in$ OSSubSort A.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. One can check that OSSubSort A is non empty.

Let us consider S_1, O_0. The functor OSSubSort O_0 yielding a set is defined by:

(Def. 8) OSSubSort $O_0 = \{x; x$ ranges over elements of SubSorts(O_0): x is an order sorted set of $S_1\}$.

The following proposition is true

(23) For every OSSubset A of O_0 holds OSSubSort $A \subseteq$ OSSubSort O_0.

Let us consider S_1, O_0. One can check that OSSubSort O_0 is non empty.

Let us consider S_1, O_0 and let e be an element of OSSubSort O_0. The functor ${}^{@}e$ yielding an OSSubset of O_0 is defined by:

(Def. 9) ${}^{@}e = e$.

Next we state two propositions:

(24) For all OSSubsets A, B of O_0 holds $B \in$ OSSubSort A iff B is operations closed and OSConstants $O_0 \subseteq B$ and $A \subseteq B$.

(25) For every OSSubset B of O_0 holds $B \in$ OSSubSort O_0 iff B is operations closed.

Let us consider S_1, O_0, let A be an OSSubset of O_0, and let s be an element of the carrier of S_1. The functor OSSubSort(A, s) yields a set and is defined by:

(Def. 10) For every set x holds $x \in$ OSSubSort(A, s) iff there exists an OSSubset B of O_0 such that $B \in$ OSSubSort A and $x = B(s)$.

We now state three propositions:

(26) For every OSSubset A of O_0 and for all sort symbols s_1, s_2 of S_1 such that $s_1 \leqslant s_2$ holds $\mathrm{OSSubSort}(A, s_2)$ is coarser than $\mathrm{OSSubSort}(A, s_1)$.

(27) For every OSSubset A of O_0 and for every sort symbol s of S_1 holds $\mathrm{OSSubSort}(A, s) \subseteq \mathrm{SubSort}(A, s)$.

(28) For every OSSubset A of O_0 and for every sort symbol s of S_1 holds (the sorts of O_0)$(s) \in \mathrm{OSSubSort}(A, s)$.

Let us consider S_1, O_0, let A be an OSSubset of O_0, and let s be a sort symbol of S_1. Note that $\mathrm{OSSubSort}(A, s)$ is non empty.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. The functor OSMSubSort A yields an OSSubset of O_0 and is defined by:

(Def. 11) For every sort symbol s of S_1 holds $(\mathrm{OSMSubSort}\,A)(s) = \bigcap \mathrm{OSSubSort}(A, s)$.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. We say that A is os-opers closed if and only if:

(Def. 12) A is operations closed.

Let us consider S_1, O_0. One can verify that there exists an OSSubset of O_0 which is os-opers closed.

Next we state several propositions:

(29) For every OSSubset A of O_0 holds $\mathrm{OSConstants}\,O_0 \cup A \subseteq \mathrm{OSMSubSort}\,A$.

(30) For every OSSubset A of O_0 such that $\mathrm{OSConstants}\,O_0 \cup A$ is non-empty holds OSMSubSort A is non-empty.

(31) Let o be an operation symbol of S_1, A be an OSSubset of O_0, and B be an OSSubset of O_0. If $B \in \mathrm{OSSubSort}\,A$, then $((\mathrm{OSMSubSort}\,A)^{\#} \cdot \text{the arity of } S_1)(o) \subseteq (B^{\#} \cdot \text{the arity of } S_1)(o)$.

(32) Let o be an operation symbol of S_1, A be an OSSubset of O_0, and B be an OSSubset of O_0. Suppose $B \in \mathrm{OSSubSort}\,A$. Then $\mathrm{rng}(\mathrm{Den}(o, O_0) \restriction ((\mathrm{OSMSubSort}\,A)^{\#} \cdot \text{the arity of } S_1)(o)) \subseteq (B \cdot \text{the result sort of } S_1)(o)$.

(33) Let o be an operation symbol of S_1 and A be an OSSubset of O_0. Then $\mathrm{rng}(\mathrm{Den}(o, O_0) \restriction ((\mathrm{OSMSubSort}\,A)^{\#} \cdot \text{the arity of } S_1)(o)) \subseteq (\mathrm{OSMSubSort}\,A \cdot \text{the result sort of } S_1)(o)$.

(34) For every OSSubset A of O_0 holds OSMSubSort A is operations closed and $A \subseteq \mathrm{OSMSubSort}\,A$.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. Note that OSMSubSort A is os-opers closed.

5. Operations on Subalgebras of an Order Sorted Algebra

Let us consider S_1, O_0 and let A be an os-opers closed OSSubset of O_0. Note that $O_0 \restriction A$ is order-sorted.

Let us consider S_1, O_0 and let O_1, O_2 be OSSubAlgebras of O_0. One can check that $O_1 \cap O_2$ is order-sorted.

Let us consider S_1, O_0 and let A be an OSSubset of O_0. The functor OSGen A yields a strict OSSubAlgebra of O_0 and is defined by the conditions (Def. 13).

(Def. 13)(i) A is an OSSubset of OSGen A, and

(ii) for every OSSubAlgebra O_1 of O_0 such that A is an OSSubset of O_1 holds OSGen A is an OSSubAlgebra of O_1.

We now state several propositions:

(35) For every OSSubset A of O_0 holds OSGen $A = O_0 \restriction$ OSMSubSort A and the sorts of OSGen A = OSMSubSort A.

(36) Let S be a non void non empty many sorted signature, U_0 be an algebra over S, and A be a subset of U_0. Then Gen(A) $= U_0 \restriction$ MSSubSort(A) and the sorts of Gen(A) = MSSubSort(A).

(37) For every OSSubset A of O_0 holds the sorts of Gen(A) $\subseteq$ the sorts of OSGen A.

(38) For every OSSubset A of O_0 holds Gen(A) is a subalgebra of OSGen A.

(39) Let O_0 be a strict order sorted algebra of S_1 and B be an OSSubset of O_0. If B = the sorts of O_0, then OSGen $B = O_0$.

(40) For every strict OSSubAlgebra O_1 of O_0 and for every OSSubset B of O_0 such that B = the sorts of O_1 holds OSGen $B = O_1$.

(41) For every non-empty order sorted algebra U_0 of S_1 and for every OSSubAlgebra U_1 of U_0 holds OSGen OSConstants $U_0 \cap U_1$ = OSGen OSConstants U_0.

Let us consider S_1, let U_0 be a non-empty order sorted algebra of S_1, and let U_1, U_2 be OSSubAlgebras of U_0. The functor $U_1 \sqcup_{os} U_2$ yielding a strict OSSubAlgebra of U_0 is defined by:

(Def. 14) For every OSSubset A of U_0 such that A = (the sorts of U_1) $\cup$ (the sorts of U_2) holds $U_1 \sqcup_{os} U_2$ = OSGen A.

One can prove the following propositions:

(42) Let U_0 be a non-empty order sorted algebra of S_1, U_1 be an OSSubAlgebra of U_0, and A, B be OSSubsets of U_0. If $B = A \cup$ the sorts of U_1, then OSGen $A \sqcup_{os} U_1$ = OSGen B.

(43) Let U_0 be a non-empty order sorted algebra of S_1, U_1 be an OSSubAlgebra of U_0, and B be an OSSubset of U_0. If B = the sorts of U_0, then OSGen $B \sqcup_{os} U_1$ = OSGen B.

(44) For every non-empty order sorted algebra U_0 of S_1 and for all OSSubAlgebras U_1, U_2 of U_0 holds $U_1 \sqcup_{os} U_2 = U_2 \sqcup_{os} U_1$.

(45) For every non-empty order sorted algebra U_0 of S_1 and for all strict OSSubAlgebras U_1, U_2 of U_0 holds $U_1 \cap (U_1 \sqcup_{os} U_2) = U_1$.

(46) For every non-empty order sorted algebra U_0 of S_1 and for all strict OSSubAlgebras U_1, U_2 of U_0 holds $U_1 \cap U_2 \sqcup_{os} U_2 = U_2$.

6. The Lattice of Subalgebras of an Order Sorted Algebra

Let us consider S_1, O_0. The functor OSSub O_0 yields a set and is defined by:

(Def. 15) For every x holds $x \in$ OSSub O_0 iff x is a strict OSSubAlgebra of O_0.

We now state the proposition

(47) OSSub $O_0 \subseteq$ Subalgebras(O_0).

Let S be an order sorted signature and let U_0 be an order sorted algebra of S. Note that OSSub U_0 is non empty.

Let us consider S_1, O_0. Then OSSub O_0 is a subset of Subalgebras(O_0).

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. The functor OSAlgJoin U_0 yields a binary operation on OSSub U_0 and is defined as follows:

(Def. 16) For all elements x, y of OSSub U_0 and for all strict OSSubAlgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds (OSAlgJoin U_0)(x, y) $= U_1 \sqcup_{os} U_2$.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. The functor OSAlgMeet U_0 yields a binary operation on OSSub U_0 and is defined as follows:

(Def. 17) For all elements x, y of OSSub U_0 and for all strict OSSubAlgebras U_1, U_2 of U_0 such that $x = U_1$ and $y = U_2$ holds (OSAlgMeet U_0)(x, y) $= U_1 \cap U_2$.

The following proposition is true

(48) For every non-empty order sorted algebra U_0 of S_1 and for all elements x, y of OSSub U_0 holds (OSAlgMeet U_0)(x, y) = (MSAlgMeet(U_0))(x, y).

In the sequel U_0 denotes a non-empty order sorted algebra of S_1.

We now state four propositions:

(49) OSAlgJoin U_0 is commutative.

(50) OSAlgJoin U_0 is associative.

(51) OSAlgMeet U_0 is commutative.

(52) OSAlgMeet U_0 is associative.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. The functor OSSubAlLattice U_0 yielding a strict lattice is defined by:

(Def. 18) OSSubAlLattice $U_0 = \langle$OSSub U_0, OSAlgJoin U_0, OSAlgMeet $U_0\rangle$.

Next we state the proposition

(53) For every non-empty order sorted algebra U_0 of S_1 holds OSSubAlLattice U_0 is bounded.

Let us consider S_1 and let U_0 be a non-empty order sorted algebra of S_1. Note that $\mathrm{OSSubAlLattice}\, U_0$ is bounded.

The following propositions are true:

(54) For every non-empty order sorted algebra U_0 of S_1 holds $\bot_{\mathrm{OSSubAlLattice}\, U_0} = \mathrm{OSGen}\,\mathrm{OSConstants}\, U_0$.

(55) Let U_0 be a non-empty order sorted algebra of S_1 and B be an OSSubset of U_0. If $B =$ the sorts of U_0, then $\top_{\mathrm{OSSubAlLattice}\, U_0} = \mathrm{OSGen}\, B$.

(56) For every strict non-empty order sorted algebra U_0 of S_1 holds $\top_{\mathrm{OSSubAlLattice}\, U_0} = U_0$.

Acknowledgments

Thanks to Joseph Goguen, for providing me with his articles on osas, and Andrzej Trybulec, for suggesting and funding this work in Bialystok.

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[2] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Formalized Mathematics*, 5(**1**):47–54, 1996.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[6] Artur Korniłowicz. Definitions and basic properties of boolean & union of many sorted sets. *Formalized Mathematics*, 5(**2**):279–281, 1996.
[7] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[9] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[10] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[11] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[12] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[13] Josef Urban. Order sorted algebras. *Formalized Mathematics*, 10(**3**):179–188, 2002.
[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[15] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

Received September 19, 2002