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Homomorphisms of Order Sorted Algebras[1]

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The notation and terminology used in this paper have been introduced in the following articles: [8], [12], [14], [15], [4], [5], [2], [1], [9], [11], [7], [10], [3], [13], and [6].

In this paper R denotes a non empty poset and S_1 denotes an order sorted signature.

Let us consider R and let F be a many sorted function indexed by the carrier of R. We say that F is order-sorted if and only if:

(Def. 1) For all elements s_1, s_2 of R such that $s_1 \leqslant s_2$ and for every set a_1 such that $a_1 \in \operatorname{dom} F(s_1)$ holds $a_1 \in \operatorname{dom} F(s_2)$ and $F(s_1)(a_1) = F(s_2)(a_1)$.

Next we state several propositions:

(1) For every set I and for every many sorted set A indexed by I holds id_A is "1-1".

(2) Let F be a many sorted function indexed by the carrier of R. Suppose F is order-sorted. Let s_1, s_2 be elements of R. If $s_1 \leqslant s_2$, then $\operatorname{dom} F(s_1) \subseteq \operatorname{dom} F(s_2)$ and $F(s_1) \subseteq F(s_2)$.

(3) Let A be an order sorted set of R, B be a non-empty order sorted set of R, and F be a many sorted function from A into B. Then F is order-sorted if and only if for all elements s_1, s_2 of R such that $s_1 \leqslant s_2$ and for every set a_1 such that $a_1 \in A(s_1)$ holds $F(s_1)(a_1) = F(s_2)(a_1)$.

(4) Let F be a many sorted function indexed by the carrier of R. Suppose F is order-sorted. Let w_1, w_2 be elements of (the carrier of R)*. If $w_1 \leqslant w_2$, then $F^{\#}(w_1) \subseteq F^{\#}(w_2)$.

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(5) For every order sorted set A of R holds id_A is order-sorted.

(6) Let A be an order sorted set of R, B, C be non-empty order sorted sets of R, F be a many sorted function from A into B, and G be a many sorted function from B into C. If F is order-sorted and G is order-sorted, then $G \circ F$ is order-sorted.

(7) Let A, B be order sorted sets of R and F be a many sorted function from A into B. If F is "1-1", onto, and order-sorted, then F^{-1} is order-sorted.

(8) Let A be an order sorted set of R and F be a many sorted function indexed by the carrier of R. If F is order-sorted, then $F^{\circ} A$ is an order sorted set of R.

Let us consider S_1 and let U_1, U_2 be order sorted algebras of S_1. We say that U_1 and U_2 are os-isomorphic if and only if:

(Def. 2) There exists a many sorted function from U_1 into U_2 which is an isomorphism of U_1 and U_2 and order-sorted.

The following propositions are true:

(9) For every order sorted algebra U_1 of S_1 holds U_1 and U_1 are os-isomorphic.

(10) Let U_1, U_2 be non-empty order sorted algebras of S_1. If U_1 and U_2 are os-isomorphic, then U_2 and U_1 are os-isomorphic.

Let us consider S_1 and let U_1, U_2 be order sorted algebras of S_1. Let us note that the predicate U_1 and U_2 are os-isomorphic is reflexive.

One can prove the following propositions:

(11) Let U_1, U_2, U_3 be non-empty order sorted algebras of S_1. Suppose U_1 and U_2 are os-isomorphic and U_2 and U_3 are os-isomorphic. Then U_1 and U_3 are os-isomorphic.

(12) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is order-sorted and a homomorphism of U_1 into U_2. Then $\mathrm{Im}\, F$ is order-sorted.

(13) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is order-sorted. Let o_1, o_2 be operation symbols of S_1. Suppose $o_1 \leqslant o_2$. Let x be an element of $\mathrm{Args}(o_1, U_1)$ and x_1 be an element of $\mathrm{Args}(o_2, U_1)$. If $x = x_1$, then $F\#x = F\#x_1$.

(14) Let U_1 be a monotone non-empty order sorted algebra of S_1, U_2 be a non-empty order sorted algebra of S_1, and F be a many sorted function from U_1 into U_2. Suppose F is order-sorted and a homomorphism of U_1 into U_2. Then $\mathrm{Im}\, F$ is order-sorted and $\mathrm{Im}\, F$ is a monotone order sorted algebra of S_1.

(15) For every monotone order sorted algebra U_1 of S_1 holds every OSSubAlgebra of U_1 is monotone.

Let us consider S_1 and let U_1 be a monotone order sorted algebra of S_1. One can check that there exists an OSSubAlgebra of U_1 which is monotone.

Let us consider S_1 and let U_1 be a monotone order sorted algebra of S_1. One can verify that every OSSubAlgebra of U_1 is monotone.

The following propositions are true:

(16) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then there exists a many sorted function G from U_1 into $\mathrm{Im}\, F$ such that $F = G$ and G is order-sorted and an epimorphism of U_1 onto $\mathrm{Im}\, F$.

(17) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then there exists a many sorted function F_1 from U_1 into $\mathrm{Im}\, F$ and there exists a many sorted function F_2 from $\mathrm{Im}\, F$ into U_2 such that

(i) F_1 is an epimorphism of U_1 onto $\mathrm{Im}\, F$,

(ii) F_2 is a monomorphism of $\mathrm{Im}\, F$ into U_2,

(iii) $F = F_2 \circ F_1$,

(iv) F_1 is order-sorted, and

(v) F_2 is order-sorted.

Let us consider S_1 and let U_1 be an order sorted algebra of S_1. Note that $\langle$the sorts of U_1, the characteristics of $U_1\rangle$ is order-sorted.

One can prove the following propositions:

(18) Let U_1 be an order sorted algebra of S_1. Then U_1 is monotone if and only if $\langle$the sorts of U_1, the characteristics of $U_1\rangle$ is monotone.

(19) Let U_1, U_2 be strict non-empty order sorted algebras of S_1. Suppose U_1 and U_2 are os-isomorphic. Then U_1 is monotone if and only if U_2 is monotone.

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