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Sequences of Metric Spaces and an Abstract Intermediate Value Theorem[1]

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Summary. Relations of convergence of real sequences and convergence of metric spaces are investigated. An abstract intermediate value theorem for two closed sets in the range is presented. At the end, it is proven that an arc connecting the west minimal point and the east maximal point in a simple closed curve must be identical to the upper arc or lower arc of the closed curve.

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The notation and terminology used here are introduced in the following papers: [21], [22], [23], [3], [4], [2], [12], [18], [6], [1], [20], [7], [5], [8], [16], [14], [13], [15], [11], [19], [17], [9], and [10].

The following propositions are true:

(1) Let R be a non empty subset of $\mathbb{R}$ and r_0 be a real number. If for every real number r such that $r \in R$ holds $r \leqslant r_0$, then $\sup R \leqslant r_0$.

(2) Let X be a non empty metric space, S be a sequence of X, and F be a subset of $X_{\rm top}$. Suppose S is convergent and for every natural number n holds $S(n) \in F$ and F is closed. Then $\lim S \in F$.

(3) Let X, Y be non empty metric spaces, f be a map from $X_{\rm top}$ into $Y_{\rm top}$, and S be a sequence of X. Then $f \cdot S$ is a sequence of Y.

(4) Let X, Y be non empty metric spaces, f be a map from $X_{\rm top}$ into $Y_{\rm top}$, S be a sequence of X, and T be a sequence of Y. If S is convergent and $T = f \cdot S$ and f is continuous, then T is convergent.

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(5) For every non empty metric space X holds every function from $\mathbb{N}$ into the carrier of X is a sequence of X.

(6) Let s be a sequence of real numbers and S be a sequence of the metric space of real numbers such that $s = S$. Then
(i) s is convergent iff S is convergent, and
(ii) if s is convergent, then $\lim s = \lim S$.

(7) Let a, b be real numbers and s be a sequence of real numbers. If $\operatorname{rng} s \subseteq [a, b]$, then s is a sequence of $[a, b]_{\rm M}$.

(8) Let a, b be real numbers and S be a sequence of $[a, b]_{\rm M}$. Suppose $a \leqslant b$. Then S is a sequence of the metric space of real numbers.

(9) Let a, b be real numbers, S_1 be a sequence of $[a, b]_{\rm M}$, and S be a sequence of the metric space of real numbers such that $S = S_1$ and $a \leqslant b$. Then
(i) S is convergent iff S_1 is convergent, and
(ii) if S is convergent, then $\lim S = \lim S_1$.

(10) Let a, b be real numbers, s be a sequence of real numbers, and S be a sequence of $[a, b]_{\rm M}$. If $S = s$ and $a \leqslant b$ and s is convergent, then S is convergent and $\lim s = \lim S$.

(11) Let a, b be real numbers, s be a sequence of real numbers, and S be a sequence of $[a, b]_{\rm M}$. If $S = s$ and $a \leqslant b$ and s is non-decreasing, then S is convergent.

(12) Let a, b be real numbers, s be a sequence of real numbers, and S be a sequence of $[a, b]_{\rm M}$. If $S = s$ and $a \leqslant b$ and s is non-increasing, then S is convergent.

(13) Let s be a sequence of real numbers and r_0 be a real number. Suppose for every natural number n holds $s(n) \leqslant r_0$ and s is convergent. Then $\lim s \leqslant r_0$.

(14) Let s be a sequence of real numbers and r_0 be a real number. Suppose for every natural number n holds $s(n) \geqslant r_0$ and s is convergent. Then $\lim s \geqslant r_0$.

(15) Let R be a non empty subset of $\mathbb{R}$. Suppose R is upper bounded. Then there exists a sequence s of real numbers such that s is non-decreasing and $\operatorname{rng} s \subseteq R$ and $\lim s = \sup R$.

(16) Let R be a non empty subset of $\mathbb{R}$. Suppose R is lower bounded. Then there exists a sequence s of real numbers such that s is non-increasing and $\operatorname{rng} s \subseteq R$ and $\lim s = \inf R$.

(17) Let X be a non empty metric space, f be a map from $\mathbb{I}$ into $X_{\rm top}$, F_1, F_2 be subsets of $X_{\rm top}$, and r_1, r_2 be real numbers. Suppose that $0 \leqslant r_1$ and $r_2 \leqslant 1$ and $r_1 \leqslant r_2$ and $f(r_1) \in F_1$ and $f(r_2) \in F_2$ and F_1 is closed and F_2 is closed and f is continuous and $F_1 \cup F_2 =$ the carrier of X. Then there exists a real number r such that $r_1 \leqslant r$ and $r \leqslant r_2$ and $f(r) \in F_1 \cap F_2$.

(18) Let n be a natural number, p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$, and P, P_1 be non empty subsets of the carrier of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2 and P_1 is an arc from p_2 to p_1 and $P_1 \subseteq P$, then $P_1 = P$.

(19) Let P, P_1 be compact non empty subsets of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed curve and P_1 is an arc from W-min P to E-max P and $P_1 \subseteq P$. Then $P_1 = \text{UpperArc}\,P$ or $P_1 = \text{LowerArc}\,P$.

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