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Armstrong's Axioms[1]

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Summary. We present a formalization of the seminal paper by W. W. Armstrong [1] on functional dependencies in relational data bases. The paper is formalized in its entirety including examples and applications. The formalization was done with a routine effort albeit some new notions were defined which simplified formulation of some theorems and proofs.

The definitive reference to the theory of relational databases is [15], where saturated sets are called closed sets. Armstrong's "axioms" for functional dependencies are still widely taught at all levels of database design, see for instance [13].

MML Identifier: `ARMSTRNG`.

The articles [21], [10], [28], [11], [24], [30], [32], [31], [18], [3], [9], [7], [26], [22], [4], [23], [25], [14], [20], [2], [5], [29], [8], [6], [17], [16], [27], [19], and [12] provide the notation and terminology for this paper.

1. Preliminaries

The following proposition is true

(1) Let B be a set. Suppose B is $\cap$-closed. Let X be a set and S be a finite family of subsets of X. If $X \in B$ and $S \subseteq B$, then $\mathrm{Intersect}(S) \in B$.

Let us observe that there exists a binary relation which is reflexive, antisymmetric, transitive, and non empty.

One can prove the following proposition

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(2) Let R be an antisymmetric transitive non empty binary relation and X be a finite subset of field R. If $X \neq \emptyset$, then there exists an element of X which is maximal w.r.t. X, R.

Let X be a set and let R be a binary relation. The functor $\mathrm{Maximals}_R(X)$ yields a subset of X and is defined by:

(Def. 1) For every set x holds $x \in \mathrm{Maximals}_R(X)$ iff x is maximal w.r.t. X, R.

Let x, X be sets. We say that x is $\cap$-irreducible in X if and only if:

(Def. 2) $x \in X$ and for all sets z, y such that $z \in X$ and $y \in X$ and $x = z \cap y$ holds $x = z$ or $x = y$.

We introduce x is $\cap$-reducible in X as an antonym of x is $\cap$-irreducible in X.

Let X be a non empty set. The functor $\cap$-Irreducibles(X) yields a subset of X and is defined by:

(Def. 3) For every set x holds $x \in \cap$-Irreducibles(X) iff x is $\cap$-irreducible in X.

The scheme *FinIntersect* deals with a non empty finite set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every set x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x]$

provided the parameters meet the following requirements:

- For every set x such that x is $\cap$-irreducible in $\mathcal{A}$ holds $\mathcal{P}[x]$, and
- For all sets x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{A}$ and $\mathcal{P}[x]$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x \cap y]$.

Next we state the proposition

(3) Let X be a non empty finite set and x be an element of X. Then there exists a non empty subset A of X such that $x = \bigcap A$ and for every set s such that $s \in A$ holds s is $\cap$-irreducible in X.

Let X be a set and let B be a family of subsets of X. We say that B is (B1) if and only if:

(Def. 4) $X \in B$.

Let B be a set. We introduce B is (B2) as a synonym of B is $\cap$-closed.

Let X be a set. Observe that there exists a family of subsets of X which is (B1) and (B2).

The following proposition is true

(4) Let X be a set and B be a non empty family of subsets of X. Suppose B is $\cap$-closed. Let x be an element of B. Suppose x is $\cap$-irreducible in B and $x \neq X$. Let S be a finite family of subsets of X. If $S \subseteq B$ and $x = \mathrm{Intersect}(S)$, then $x \in S$.

Let X, D be non empty sets and let n be a natural number. Observe that every function from X into D^n is finite sequence yielding.

Let f be a finite sequence yielding function and let x be a set. Note that $f(x)$ is finite sequence-like.

Let n be a natural number and let p, q be n-tuples of *Boolean*. The functor $p \wedge q$ yielding a n-tuple of *Boolean* is defined as follows:

(Def. 5) For every set i such that $i \in \mathrm{Seg}\, n$ holds $(p \wedge q)(i) = p_i \wedge q_i$.

Let us notice that the functor $p \wedge q$ is commutative.

One can prove the following propositions:

(5) For every natural number n and for every n-tuple p of *Boolean* holds $(n\text{-BinarySequence}(0)) \wedge p = n\text{-BinarySequence}(0)$.

(6) For every natural number n and for every n-tuple p of *Boolean* holds $\neg(n\text{-BinarySequence}(0)) \wedge p = p$.

(7) For every natural number i holds $(i+1)\text{-BinarySequence}(2^i) = \underbrace{\langle 0, \ldots, 0 \rangle}_{i} \frown \langle 1 \rangle$.

(8) Let n, i be natural numbers. Suppose $i < n$. Then $(n\text{-BinarySequence}(2^i))(i+1) = 1$ and for every natural number j such that $j \in \mathrm{Seg}\, n$ and $j \neq i+1$ holds $(n\text{-BinarySequence}(2^i))(j) = 0$.

2. The Relational Model of Data

We consider DB-relationships as systems
⟨ attributes, domains, a relationship ⟩,
where the attributes constitute a finite non empty set, the domains constitute a non-empty many sorted set indexed by the attributes, and the relationship is a subset of $\prod$ the domains.

3. Dependency Structures

Let X be a set.

(Def. 6) A binary relation on 2^X is said to be a relation on subsets of X.

We introduce dependency set of X as a synonym of a relation on subsets of X.

Let X be a set. Observe that there exists a dependency set of X which is non empty and finite.

Let X be a set. The functor dependencies(X) yields a dependency set of X and is defined by:

(Def. 7) $\mathrm{dependencies}(X) = [\!:\, 2^X, 2^X \,:\!]$.

Let X be a set. Observe that dependencies(X) is non empty. A dependency of X is an element of dependencies(X).

Let X be a set and let F be a non empty dependency set of X. We see that the element of F is a dependency of X.

The following three propositions are true:

(9) For all sets X, x holds $x \in$ dependencies(X) iff there exist subsets a, b of X such that $x = \langle a, b \rangle$.

(10) For all sets X, x and for every dependency set F of X such that $x \in F$ there exist subsets a, b of X such that $x = \langle a, b \rangle$.

(11) For every set X and for every dependency set F of X holds every subset of F is a dependency set of X.

Let R be a DB-relationship and let A, B be subsets of the attributes of R. The predicate $A \to_R B$ is defined by:

(Def. 8) For all elements f, g of the relationship of R such that $f \restriction A = g \restriction A$ holds $f \restriction B = g \restriction B$.

We introduce (A, B) holds in R as a synonym of $A \to_R B$.

In the sequel R denotes a DB-relationship and A, B denote subsets of the attributes of R.

Let us consider R. The functor dependency-structure(R) yields a dependency set of the attributes of R and is defined as follows:

(Def. 9) dependency-structure(R) $= \{\langle A, B \rangle : A \to_R B\}$.

One can prove the following proposition

(12) For every DB-relationship R and for all subsets A, B of the attributes of R holds $\langle A, B \rangle \in$ dependency-structure(R) iff $A \to_R B$.

4. Full Families of Dependencies

Let X be a set and let P, Q be dependencies of X. The predicate $P \geqslant Q$ is defined by:

(Def. 10) $P_{\mathbf{1}} \subseteq Q_{\mathbf{1}}$ and $Q_{\mathbf{2}} \subseteq P_{\mathbf{2}}$.

Let us note that the predicate $P \geqslant Q$ is reflexive. We introduce $Q \leqslant P$ and also P is at least as informative as Q, as synonyms of $P \geqslant Q$.

The following propositions are true:

(13) For every set X and for all dependencies P, Q of X such that $P \leqslant Q$ and $Q \leqslant P$ holds $P = Q$.

(14) For every set X and for all dependencies P, Q, S of X such that $P \leqslant Q$ and $Q \leqslant S$ holds $P \leqslant S$.

Let X be a set and let A, B be subsets of X. Then $\langle A, B \rangle$ is a dependency of X.

We now state the proposition

(15) For every set X and for all subsets A, B, A', B' of X holds $\langle A, B \rangle \geqslant \langle A', B' \rangle$ iff $A \subseteq A'$ and $B' \subseteq B$.

Let X be a set. The functor Dependencies-Order X yielding a binary relation on dependencies(X) is defined as follows:

(Def. 11) Dependencies-Order $X = \{\langle P, Q \rangle; P$ ranges over dependencies of X, Q ranges over dependencies of X: $P \leqslant Q\}$.

We now state four propositions:

(16) For all sets X, x holds $x \in$ Dependencies-Order X iff there exist dependencies P, Q of X such that $x = \langle P, Q \rangle$ and $P \leqslant Q$.

(17) For every set X holds dom Dependencies-Order $X = [: 2^X, 2^X :]$.

(18) For every set X holds rng Dependencies-Order $X = [: 2^X, 2^X :]$.

(19) For every set X holds field Dependencies-Order $X = [: 2^X, 2^X :]$.

Let X be a set. Note that Dependencies-Order X is non empty and Dependencies-Order X is ordering.

Let X be a set and let F be a dependency set of X. We say that F is (F1) if and only if:

(Def. 12) For every subset A of X holds $\langle A, A \rangle \in F$.

We introduce F is (DC2) as a synonym of F is (F1). We introduce F is (F2) and F is (DC1) as synonyms of F is transitive.

The following proposition is true

(20) Let X be a set and F be a dependency set of X. Then F is (F2) if and only if for all subsets A, B, C of X such that $\langle A, B \rangle \in F$ and $\langle B, C \rangle \in F$ holds $\langle A, C \rangle \in F$.

Let X be a set and let F be a dependency set of X. We say that F is (F3) if and only if:

(Def. 13) For all subsets A, B, A', B' of X such that $\langle A, B \rangle \in F$ and $\langle A, B \rangle \geqslant \langle A', B' \rangle$ holds $\langle A', B' \rangle \in F$.

We say that F is (F4) if and only if:

(Def. 14) For all subsets A, B, A', B' of X such that $\langle A, B \rangle \in F$ and $\langle A', B' \rangle \in F$ holds $\langle A \cup A', B \cup B' \rangle \in F$.

The following proposition is true

(21) For every set X holds dependencies(X) is (F1), (F2), (F3), and (F4).

Let X be a set. Observe that there exists a dependency set of X which is (F1), (F2), (F3), (F4), and non empty.

Let X be a set and let F be a dependency set of X. We say that F is full family if and only if:

(Def. 15) F is (F1), (F2), (F3), and (F4).

Let X be a set. One can verify that there exists a dependency set of X which is full family.

Let X be a set. A Full family of X is a full family dependency set of X.

We now state the proposition

(22) For every finite set X holds every dependency set of X is finite.

Let X be a finite set. Observe that there exists a Full family of X which is finite and every dependency set of X is finite.

Let X be a set. Note that every dependency set of X which is full family is also (F1), (F2), (F3), and (F4) and every dependency set of X which is (F1), (F2), (F3), and (F4) is also full family.

Let X be a set and let F be a dependency set of X. We say that F is (DC3) if and only if:

(Def. 16) For all subsets A, B of X such that $B \subseteq A$ holds $\langle A, B \rangle \in F$.

Let X be a set. Observe that every dependency set of X which is (F1) and (F3) is also (DC3) and every dependency set of X which is (DC3) and (F2) is also (F1) and (F3).

Let X be a set. Observe that there exists a dependency set of X which is (DC3), (F2), (F4), and non empty.

We now state two propositions:

(23) For every set X and for every dependency set F of X such that F is (DC3) and (F2) holds F is (F1) and (F3).

(24) For every set X and for every dependency set F of X such that F is (F1) and (F3) holds F is (DC3).

Let X be a set. Observe that every dependency set of X which is (F1) is also non empty.

The following propositions are true:

(25) For every DB-relationship R holds dependency-structure(R) is full family.

(26) Let X be a set and K be a subset of X. Then $\{\langle A, B \rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \ \vee \ B \subseteq A\}$ is a Full family of X.

5. Maximal Elements of Full Families

Let X be a set and let F be a dependency set of X. The functor Maximals(F) yielding a dependency set of X is defined as follows:

(Def. 17) $\mathrm{Maximals}(F) = \mathrm{Maximals}_{\text{Dependencies-Order}\,X}(F)$.

We now state the proposition

(27) For every set X and for every dependency set F of X holds $\mathrm{Maximals}(F) \subseteq F$.

Let X be a set, let F be a dependency set of X, and let x, y be sets. The predicate $x \nearrow_F y$ is defined as follows:

(Def. 18) $\langle x, y \rangle \in \mathrm{Maximals}(F)$.

One can prove the following two propositions:

(28) Let X be a finite set, P be a dependency of X, and F be a dependency set of X. If $P \in F$, then there exist subsets A, B of X such that $\langle A, B\rangle \in \mathrm{Maximals}(F)$ and $\langle A, B\rangle \geqslant P$.

(29) Let X be a set, F be a dependency set of X, and A, B be subsets of X. Then $A \nearrow_F B$ if and only if the following conditions are satisfied:

(i) $\langle A, B\rangle \in F$, and

(ii) it is not true that there exist subsets A', B' of X such that $\langle A', B'\rangle \in F$ and $\langle A, B\rangle \leqslant \langle A', B'\rangle$ with $A \neq A'$ or $B \neq B'$.

Let X be a set and let M be a dependency set of X. We say that M is (M1) if and only if:

(Def. 19) For every subset A of X there exist subsets A', B' of X such that $\langle A', B'\rangle \geqslant \langle A, A\rangle$ and $\langle A', B'\rangle \in M$.

We say that M is (M2) if and only if:

(Def. 20) For all subsets A, B, A', B' of X such that $\langle A, B\rangle \in M$ and $\langle A', B'\rangle \in M$ and $\langle A, B\rangle \geqslant \langle A', B'\rangle$ holds $A = A'$ and $B = B'$.

We say that M is (M3) if and only if:

(Def. 21) For all subsets A, B, A', B' of X such that $\langle A, B\rangle \in M$ and $\langle A', B'\rangle \in M$ and $A' \subseteq B$ holds $B' \subseteq B$.

We now state two propositions:

(30) For every finite non empty set X and for every Full family F of X holds $\mathrm{Maximals}(F)$ is (M1), (M2), and (M3).

(31) Let X be a finite set and M, F be dependency sets of X. Suppose that

(i) M is (M1), (M2), and (M3), and

(ii) $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $\bigvee_{A',B' : \text{subset of } X} (\langle A', B'\rangle \geqslant \langle A, B\rangle \ \wedge\ \langle A', B'\rangle \in M)\}$.

Then $M = \mathrm{Maximals}(F)$ and F is full family and for every Full family G of X such that $M = \mathrm{Maximals}(G)$ holds $G = F$.

Let X be a non empty finite set and let F be a Full family of X. Note that $\mathrm{Maximals}(F)$ is non empty.

Next we state the proposition

(32) Let X be a finite set, F be a dependency set of X, and K be a subset of X. Suppose $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \vee B \subseteq A\}$. Then $\{\langle K, X\rangle\} \cup \{\langle A, A\rangle; A$ ranges over subsets of X: $K \not\subseteq A\} = \mathrm{Maximals}(F)$.

6. Saturated Subsets of Attributes

Let X be a set and let F be a dependency set of X.

The functor saturated-subsets(F) yields a family of subsets of X and is defined as follows:

(Def. 22) saturated-subsets(F) =
$\{B; B$ ranges over subsets of X: $\bigvee_{A\,:\,\text{subset of } X}\ A \nearrow_F B\}$.

We introduce closed-attribute-subset(F) as a synonym of saturated-subsets(F).

Let X be a set and let F be a finite dependency set of X. Observe that saturated-subsets(F) is finite.

Next we state two propositions:

(33) Let X, x be sets and F be a dependency set of X. Then $x \in$ saturated-subsets(F) if and only if there exist subsets B, A of X such that $x = B$ and $A \nearrow_F B$.

(34) For every finite non empty set X and for every Full family F of X holds saturated-subsets(F) is (B1) and (B2).

Let X be a set and let B be a set. The functor (B)-enclosed in X yields a dependency set of X and is defined as follows:

(Def. 23) (B)-enclosed in X = $\{\langle a, b\rangle; a$ ranges over subsets of X, b ranges over subsets of X: $\bigwedge_{c\,:\,\text{set}} (c \in B \ \wedge\ a \subseteq c \ \Rightarrow\ b \subseteq c)\}$.

The following three propositions are true:

(35) For every set X and for every family B of subsets of X and for every dependency set F of X holds (B)-enclosed in X is full family.

(36) For every finite non empty set X and for every family B of subsets of X holds $B \subseteq$ saturated-subsets((B)-enclosed in X).

(37) Let X be a finite non empty set and B be a family of subsets of X. Suppose B is (B1) and (B2). Then $B =$ saturated-subsets((B)-enclosed in X) and for every Full family G of X such that $B =$ saturated-subsets(G) holds $G =$ (B)-enclosed in X.

Let X be a set and let F be a dependency set of X. The functor (F)-enclosure yielding a family of subsets of X is defined as follows:

(Def. 24) (F)-enclosure = $\{b; b$ ranges over subsets of X: $\bigwedge_{A,B\,:\,\text{subset of } X} (\langle A, B\rangle \in F \ \wedge\ A \subseteq b \ \Rightarrow\ B \subseteq b)\}$.

We now state two propositions:

(38) For every finite non empty set X and for every dependency set F of X holds (F)-enclosure is (B1) and (B2).

(39) Let X be a finite non empty set and F be a dependency set of X. Then $F \subseteq$ ((F)-enclosure)-enclosed in X and for every dependency set G of X such that $F \subseteq G$ and G is full family holds ((F)-enclosure)-enclosed in $X \subseteq G$.

Let X be a finite non empty set and let F be a dependency set of X. The functor dependency-closure(F) yields a Full family of X and is defined by:

(Def. 25) $F \subseteq$ dependency-closure(F) and for every dependency set G of X such that $F \subseteq G$ and G is full family holds dependency-closure(F) $\subseteq G$.

Next we state four propositions:

(40) For every finite non empty set X and for every dependency set F of X holds dependency-closure(F) = ((F)-enclosure)-enclosed in X.

(41) Let X be a set, K be a subset of X, and B be a family of subsets of X. If $B = \{X\} \cup \{A; A$ ranges over subsets of X: $K \not\subseteq A\}$, then B is (B1) and (B2).

(42) Let X be a finite non empty set, F be a dependency set of X, and K be a subset of X. Suppose $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \ \vee \ B \subseteq A\}$. Then $\{X\} \cup \{B; B$ ranges over subsets of X: $K \not\subseteq B\}$ = saturated-subsets(F).

(43) Let X be a finite set, F be a dependency set of X, and K be a subset of X. Suppose $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \ \vee \ B \subseteq A\}$. Then $\{X\} \cup \{B; B$ ranges over subsets of X: $K \not\subseteq B\}$ = saturated-subsets(F).

Let X, G be sets and let B be a family of subsets of X. We say that G is generator set of B if and only if:

(Def. 26) $G \subseteq B$ and $B = \{$Intersect$(S); S$ ranges over families of subsets of X: $S \subseteq G\}$.

We now state four propositions:

(44) For every finite non empty set X holds every family G of subsets of X is generator set of saturated-subsets((G)-enclosed in X).

(45) Let X be a finite non empty set and F be a Full family of X. Then there exists a family G of subsets of X such that G is generator set of saturated-subsets(F) and F = (G)-enclosed in X.

(46) Let X be a set and B be a non empty finite family of subsets of X. If B is (B1) and (B2), then $\cap$-Irreducibles(B) is generator set of B.

(47) Let X, G be sets and B be a non empty finite family of subsets of X. If B is (B1) and (B2) and G is generator set of B, then $\cap$-Irreducibles$(B) \subseteq G \cup \{X\}$.

7. Justification of the Axioms

One can prove the following proposition

(48) Let X be a non empty finite set and F be a Full family of X. Then there exists a DB-relationship R such that the attributes of $R = X$ and for every element a of X holds (the domains of R)(a) $= \mathbb{Z}$ and F = dependency-structure(R).

8. Structure of the Family of Candidate Keys

Let X be a set and let F be a dependency set of X.

The functor candidate-keys(F) yields a family of subsets of X and is defined by:

(Def. 27) candidate-keys(F) $= \{A; A$ ranges over subsets of X: $\langle A, X\rangle \in$ Maximals$(F)\}$.

One can prove the following proposition

(49) Let X be a finite set, F be a dependency set of X, and K be a subset of X. Suppose $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \lor B \subseteq A\}$. Then candidate-keys($F$) $= \{K\}$.

Let X be a set. We introduce X is (C1) as an antonym of X is empty.

Let X be a set. We say that X is without proper subsets if and only if:

(Def. 28) For all sets x, y such that $x \in X$ and $y \in X$ and $x \subseteq y$ holds $x = y$.

We introduce X is (C2) as a synonym of X is without proper subsets.

We now state four propositions:

(50) For every DB-relationship R holds candidate-keys(dependency-structure(R)) is (C1) and (C2).

(51) Let X be a finite set and C be a family of subsets of X. If C is (C1) and (C2), then there exists a Full family F of X such that $C =$ candidate-keys(F).

(52) Let X be a finite set, C be a family of subsets of X, and B be a set. Suppose C is (C1) and (C2) and $B = \{b; b$ ranges over subsets of X: $\bigwedge_{K\,:\,\text{subset of } X} (K \in C \Rightarrow K \not\subseteq b)\}$. Then $C =$ candidate-keys((B)-enclosed in X).

(53) Let X be a non empty finite set and C be a family of subsets of X. Suppose C is (C1) and (C2). Then there exists a DB-relationship R such that the attributes of $R = X$ and $C =$ candidate-keys(dependency-structure(R)).

9. Applications

Let X be a set and let F be a dependency set of X. We say that F is (DC4) if and only if:

(Def. 29) For all subsets A, B, C of X such that $\langle A, B\rangle \in F$ and $\langle A, C\rangle \in F$ holds $\langle A, B \cup C\rangle \in F$.

We say that F is (DC5) if and only if:

(Def. 30) For all subsets A, B, C, D of X such that $\langle A, B\rangle \in F$ and $\langle B \cup C, D\rangle \in F$ holds $\langle A \cup C, D\rangle \in F$.

We say that F is (DC6) if and only if:

(Def. 31) For all subsets A, B, C of X such that $\langle A, B\rangle \in F$ holds $\langle A \cup C, B\rangle \in F$.

One can prove the following propositions:

(54) Let X be a set and F be a dependency set of X. Then F is (F1), (F2), (F3), and (F4) if and only if F is (F2), (DC3), and (F4).

(55) Let X be a set and F be a dependency set of X. Then F is (F1), (F2), (F3), and (F4) if and only if F is (DC1), (DC3), and (DC4).

(56) Let X be a set and F be a dependency set of X. Then F is (F1), (F2), (F3), and (F4) if and only if F is (DC2), (DC5), and (DC6).

Let X be a set and let F be a dependency set of X.

The functor characteristic(F) is defined as follows:

(Def. 32) characteristic(F) $= \{A; A$ ranges over subsets of X: $\bigvee_{a,b\,:\,\text{subset of }X} (\langle a, b\rangle \in F \ \wedge\ a \subseteq A \ \wedge\ b \not\subseteq A)\}$.

Next we state several propositions:

(57) Let X, A be sets and F be a dependency set of X. Suppose $A \in$ characteristic(F). Then A is a subset of X and there exist subsets a, b of X such that $\langle a, b\rangle \in F$ and $a \subseteq A$ and $b \not\subseteq A$.

(58) Let X be a set, A be a subset of X, and F be a dependency set of X. If there exist subsets a, b of X such that $\langle a, b\rangle \in F$ and $a \subseteq A$ and $b \not\subseteq A$, then $A \in$ characteristic(F).

(59) Let X be a finite non empty set and F be a dependency set of X. Then
 (i) for all subsets A, B of X holds $\langle A, B\rangle \in$ dependency-closure(F) iff for every subset a of X such that $A \subseteq a$ and $B \not\subseteq a$ holds $a \in$ characteristic(F), and
 (ii) saturated-subsets(dependency-closure(F)) $= 2^X \setminus$ characteristic(F).

(60) For every finite non empty set X and for all dependency sets F, G of X such that characteristic(F) $=$ characteristic(G) holds dependency-closure(F) $=$ dependency-closure(G).

(61) For every non empty finite set X and for every dependency set F of X holds characteristic(F) $=$ characteristic(dependency-closure(F)).

Let A be a set, let K be a set, and let F be a dependency set of A. We say that K is prime implicant of F with no complemented variables if and only if the conditions (Def. 33) are satisfied.

(Def. 33)(i) For every subset a of A such that $K \subseteq a$ and $a \neq A$ holds $a \in$ characteristic(F), and
 (ii) for every set k such that $k \subset K$ there exists a subset a of A such that $k \subseteq a$ and $a \neq A$ and $a \notin$ characteristic(F).

The following proposition is true

(62) Let X be a finite non empty set, F be a dependency set of X, and K be a subset of X. Then $K \in$ candidate-keys(dependency-closure(F)) if and only if K is prime implicant of F with no complemented variables.

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