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Subspaces and Cosets of Subspace of Real Unitary Space

Noboru Endou
Gifu National College of Technology

Takashi Mitsuishi
Miyagi University

Yasunari Shidama
Shinshu University
Nagano

Summary. In this article, subspace and the coset of subspace of real unitary space are defined. And we discuss some of their fundamental properties.

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The articles [6], [3], [10], [7], [1], [11], [2], [5], [9], [8], and [4] provide the notation and terminology for this paper.

1. Definition and Axioms of the Subspace of Real Unitary Space

Let V be a real unitary space. A real unitary space is said to be a subspace of V if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the zero of it = the zero of V,

(iii) the addition of it = (the addition of V)$\restriction$[: the carrier of it, the carrier of it :],

(iv) the external multiplication of it = (the external multiplication of V)$\restriction$[: $\mathbb{R}$, the carrier of it :], and

(v) the scalar product of it = (the scalar product of V)$\restriction$[: the carrier of it, the carrier of it :].

We now state a number of propositions:

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(1) Let V be a real unitary space, W_1, W_2 be subspaces of V, and x be a set. If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(2) For every real unitary space V and for every subspace W of V and for every set x such that $x \in W$ holds $x \in V$.

(3) For every real unitary space V and for every subspace W of V holds every vector of W is a vector of V.

(4) For every real unitary space V and for every subspace W of V holds $0_W = 0_V$.

(5) For every real unitary space V and for all subspaces W_1, W_2 of V holds $0_{(W_1)} = 0_{(W_2)}$.

(6) Let V be a real unitary space, W be a subspace of V, u, v be vectors of V, and w_1, w_2 be vectors of W. If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(7) Let V be a real unitary space, W be a subspace of V, v be a vector of V, w be a vector of W, and a be a real number. If $w = v$, then $a \cdot w = a \cdot v$.

(8) Let V be a real unitary space, W be a subspace of V, v_1, v_2 be vectors of V, and w_1, w_2 be vectors of W. If $w_1 = v_1$ and $w_2 = v_2$, then $(w_1|w_2) = (v_1|v_2)$.

(9) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and w be a vector of W. If $w = v$, then $-v = -w$.

(10) Let V be a real unitary space, W be a subspace of V, u, v be vectors of V, and w_1, w_2 be vectors of W. If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(11) For every real unitary space V and for every subspace W of V holds $0_V \in W$.

(12) For every real unitary space V and for all subspaces W_1, W_2 of V holds $0_{(W_1)} \in W_2$.

(13) For every real unitary space V and for every subspace W of V holds $0_W \in V$.

(14) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $u \in W$ and $v \in W$, then $u + v \in W$.

(15) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $v \in W$, then $a \cdot v \in W$.

(16) For every real unitary space V and for every subspace W of V and for every vector v of V such that $v \in W$ holds $-v \in W$.

(17) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $u \in W$ and $v \in W$, then $u - v \in W$.

(18) Let V be a real unitary space, V_1 be a subset of the carrier of V, D be a non empty set, d_1 be an element of D, A be a binary operation on D, M be a function from $[:\mathbb{R}, D:]$ into D, and S be a function from $[:D, D:]$ into $\mathbb{R}$. Suppose that

(i) $V_1 = D$,
(ii) $d_1 = 0_V$,
(iii) $A =$ (the addition of V)$\restriction [\!: V_1, V_1 :\!]$,
(iv) $M =$ (the external multiplication of V)$\restriction [\!: \mathbb{R}, V_1 :\!]$, and
(v) $S =$ (the scalar product of V)$\restriction [\!: V_1, V_1 :\!]$.

Then $\langle D, d_1, A, M, S \rangle$ is a subspace of V.

(19) Every real unitary space V is a subspace of V.

(20) For all strict real unitary spaces V, X such that V is a subspace of X and X is a subspace of V holds $V = X$.

(21) Let V, X, Y be real unitary spaces. Suppose V is a subspace of X and X is a subspace of Y. Then V is a subspace of Y.

(22) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose the carrier of $W_1 \subseteq$ the carrier of W_2. Then W_1 is a subspace of W_2.

(23) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose that for every vector v of V such that $v \in W_1$ holds $v \in W_2$. Then W_1 is a subspace of W_2.

Let V be a real unitary space. Observe that there exists a subspace of V which is strict.

Next we state several propositions:

(24) Let V be a real unitary space and W_1, W_2 be strict subspaces of V. If the carrier of $W_1 =$ the carrier of W_2, then $W_1 = W_2$.

(25) Let V be a real unitary space and W_1, W_2 be strict subspaces of V. If for every vector v of V holds $v \in W_1$ iff $v \in W_2$, then $W_1 = W_2$.

(26) Let V be a strict real unitary space and W be a strict subspace of V. If the carrier of $W =$ the carrier of V, then $W = V$.

(27) Let V be a strict real unitary space and W be a strict subspace of V. If for every vector v of V holds $v \in W$ iff $v \in V$, then $W = V$.

(28) Let V be a real unitary space, W be a subspace of V, and V_1 be a subset of the carrier of V. If the carrier of $W = V_1$, then V_1 is linearly closed.

(29) Let V be a real unitary space, W be a subspace of V, and V_1 be a subset of the carrier of V. Suppose $V_1 \neq \emptyset$ and V_1 is linearly closed. Then there exists a strict subspace W of V such that $V_1 =$ the carrier of W.

2. Definition of Zero Subspace and Improper Subspace of Real Unitary Space

Let V be a real unitary space. The functor $\mathbf{0}_V$ yields a strict subspace of V and is defined by:

(Def. 2) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let V be a real unitary space. The functor Ω_V yielding a strict subspace of V is defined by:

(Def. 3) $\Omega_V =$ the unitary space structure of V.

3. Theorems of Zero Subspace and Improper Subspace

Next we state several propositions:

(30) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_W = \mathbf{0}_V$.

(31) For every real unitary space V and for all subspaces W_1, W_2 of V holds $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(32) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_W$ is a subspace of V.

(33) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_V$ is a subspace of W.

(34) For every real unitary space V and for all subspaces W_1, W_2 of V holds $\mathbf{0}_{(W_1)}$ is a subspace of W_2.

(35) Every strict real unitary space V is a subspace of Ω_V.

4. The Cosets of Subspace of Real Unitary Space

Let V be a real unitary space, let v be a vector of V, and let W be a subspace of V. The functor $v + W$ yields a subset of the carrier of V and is defined as follows:

(Def. 4) $v + W = \{v + u; u$ ranges over vectors of V: $u \in W\}$.

Let V be a real unitary space and let W be a subspace of V. A subset of the carrier of V is said to be a coset of W if:

(Def. 5) There exists a vector v of V such that it $= v + W$.

5. Theorems of the Cosets

We now state a number of propositions:

(36) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $0_V \in v + W$ if and only if $v \in W$.

(37) For every real unitary space V and for every subspace W of V and for every vector v of V holds $v \in v + W$.

(38) For every real unitary space V and for every subspace W of V holds $0_V + W =$ the carrier of W.

(39) For every real unitary space V and for every vector v of V holds $v + \mathbf{0}_V = \{v\}$.

(40) For every real unitary space V and for every vector v of V holds $v + \Omega_V =$ the carrier of V.

(41) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $0_V \in v + W$ if and only if $v + W =$ the carrier of W.

(42) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $v \in W$ if and only if $v + W =$ the carrier of W.

(43) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $v \in W$, then $a \cdot v + W =$ the carrier of W.

(44) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $a \neq 0$ and $a \cdot v + W =$ the carrier of W, then $v \in W$.

(45) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $v \in W$ if and only if $-v + W =$ the carrier of W.

(46) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in W$ if and only if $v + W = v + u + W$.

(47) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in W$ if and only if $v + W = (v - u) + W$.

(48) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $v \in u + W$ if and only if $u + W = v + W$.

(49) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $v + W = -v + W$ if and only if $v \in W$.

(50) Let V be a real unitary space, W be a subspace of V, and u, v_1, v_2 be vectors of V. If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.

(51) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $u \in v + W$ and $u \in -v + W$, then $v \in W$.

(52) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $a \neq 1$ and $a \cdot v \in v + W$, then $v \in W$.

(53) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $v \in W$, then $a \cdot v \in v + W$.

(54) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $-v \in v + W$ if and only if $v \in W$.

(55) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u + v \in v + W$ if and only if $u \in W$.

(56) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $v - u \in v + W$ if and only if $u \in W$.

(57) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in v + W$ if and only if there exists a vector v_1 of V such that

$v_1 \in W$ and $u = v + v_1$.

(58) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in v+W$ if and only if there exists a vector v_1 of V such that $v_1 \in W$ and $u = v - v_1$.

(59) Let V be a real unitary space, W be a subspace of V, and v_1, v_2 be vectors of V. Then there exists a vector v of V such that $v_1 \in v + W$ and $v_2 \in v + W$ if and only if $v_1 - v_2 \in W$.

(60) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $v+W = u+W$, then there exists a vector v_1 of V such that $v_1 \in W$ and $v + v_1 = u$.

(61) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $v+W = u+W$, then there exists a vector v_1 of V such that $v_1 \in W$ and $v - v_1 = u$.

(62) Let V be a real unitary space, W_1, W_2 be strict subspaces of V, and v be a vector of V. Then $v + W_1 = v + W_2$ if and only if $W_1 = W_2$.

(63) Let V be a real unitary space, W_1, W_2 be strict subspaces of V, and u, v be vectors of V. If $v + W_1 = u + W_2$, then $W_1 = W_2$.

(64) Let V be a real unitary space, W be a subspace of V, and C be a coset of W. Then C is linearly closed if and only if $C =$ the carrier of W.

(65) Let V be a real unitary space, W_1, W_2 be strict subspaces of V, C_1 be a coset of W_1, and C_2 be a coset of W_2. If $C_1 = C_2$, then $W_1 = W_2$.

(66) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and v be a vector of V. Then $\{v\}$ is a coset of $\mathbf{0}_V$.

(67) Let V be a real unitary space, W be a subspace of V, V_1 be a subset of the carrier of V, and v be a vector of V. If V_1 is a coset of $\mathbf{0}_V$, then there exists a vector v of V such that $V_1 = \{v\}$.

(68) For every real unitary space V and for every subspace W of V holds the carrier of W is a coset of W.

(69) For every real unitary space V holds the carrier of V is a coset of Ω_V.

(70) Let V be a real unitary space, W be a subspace of V, and V_1 be a subset of the carrier of V. If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of V.

(71) Let V be a real unitary space, W be a subspace of V, and C be a coset of W. Then $0_V \in C$ if and only if $C =$ the carrier of W.

(72) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and u be a vector of V. Then $u \in C$ if and only if $C = u + W$.

(73) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and u, v be vectors of V. If $u \in C$ and $v \in C$, then there exists a vector v_1 of V such that $v_1 \in W$ and $u + v_1 = v$.

(74) Let V be a real unitary space, W be a subspace of V, C be a coset of W,

and u, v be vectors of V. If $u \in C$ and $v \in C$, then there exists a vector v_1 of V such that $v_1 \in W$ and $u - v_1 = v$.

(75) Let V be a real unitary space, W be a subspace of V, and v_1, v_2 be vectors of V. Then there exists a coset C of W such that $v_1 \in C$ and $v_2 \in C$ if and only if $v_1 - v_2 \in W$.

(76) Let V be a real unitary space, W be a subspace of V, u be a vector of V, and B, C be cosets of W. If $u \in B$ and $u \in C$, then $B = C$.

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