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Topology of Real Unitary Space

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Summary. In this article we introduce three subjects in real unitary space: parallelism of subsets, orthogonality of subsets and topology of the space. In particular, to introduce the topology of real unitary space, we discuss the metric topology which is induced by the inner product in the space. As the result, we are able to discuss some topological subjects on real unitary space.

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The articles [8], [12], [3], [5], [4], [11], [10], [9], [6], [7], [2], and [1] provide the terminology and notation for this paper.

1. Parallelism of Subspaces

Let V be a non empty RLS structure and let M, N be Affine subsets of V. We say that M is parallel to N if and only if:

(Def. 1) There exists a vector v of V such that $M = N + \{v\}$.

One can prove the following propositions:

(1) For every right zeroed non empty RLS structure V holds every Affine subset M of V is parallel to M.

(2) Let V be an add-associative right zeroed right complementable non empty RLS structure and M, N be Affine subsets of V. If M is parallel to N, then N is parallel to M.

(3) Let V be an Abelian add-associative right zeroed right complementable non empty RLS structure and M, L, N be Affine subsets of V. If M is parallel to L and L is parallel to N, then M is parallel to N.

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Let V be a non empty loop structure and let M, N be subsets of the carrier of V. The functor $M - N$ yields a subset of V and is defined as follows:

(Def. 2) $M - N = \{u - v; u$ ranges over elements of the carrier of V, v ranges over elements of the carrier of V: $u \in M \ \wedge \ v \in N\}$.

Next we state a number of propositions:

(4) For every real linear space V and for all Affine subsets M, N of V holds $M - N$ is Affine.

(5) For every non empty loop structure V and for all subsets M, N of V such that M is empty or N is empty holds $M + N$ is empty.

(6) For every non empty loop structure V and for all non empty subsets M, N of V holds $M + N$ is non empty.

(7) For every non empty loop structure V and for all subsets M, N of V such that M is empty or N is empty holds $M - N$ is empty.

(8) For every non empty loop structure V and for all non empty subsets M, N of V holds $M - N$ is non empty.

(9) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure, M, N be subsets of V, and v be an element of the carrier of V. Then $M = N + \{v\}$ if and only if $M - \{v\} = N$.

(10) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure, M, N be subsets of V, and v be an element of the carrier of V. If $v \in N$, then $M + \{v\} \subseteq M + N$.

(11) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure, M, N be subsets of V, and v be an element of the carrier of V. If $v \in N$, then $M - \{v\} \subseteq M - N$.

(12) For every real linear space V and for every non empty subset M of V holds $0_V \in M - M$.

(13) Let V be a real linear space, M be a non empty Affine subset of V, and v be a vector of V. If M is Subspace-like and $v \in M$, then $M + \{v\} \subseteq M$.

(14) Let V be a real linear space, M be a non empty Affine subset of V, and N_1, N_2 be non empty Affine subsets of V. Suppose N_1 is Subspace-like and N_2 is Subspace-like and M is parallel to N_1 and parallel to N_2. Then $N_1 = N_2$.

(15) Let V be a real linear space, M be a non empty Affine subset of V, and v be a vector of V. If $v \in M$, then $0_V \in M - \{v\}$.

(16) Let V be a real linear space, M be a non empty Affine subset of V, and v be a vector of V. Suppose $v \in M$. Then there exists a non empty Affine subset N of V such that $N = M - \{v\}$ and M is parallel to N and N is Subspace-like.

(17) Let V be a real linear space, M be a non empty Affine subset of V, and

u, v be vectors of V. If $u \in M$ and $v \in M$, then $M - \{v\} = M - \{u\}$.

(18) For every real linear space V and for every non empty Affine subset M of V holds $M - M = \bigcup\{M - \{v\}; v$ ranges over vectors of V: $v \in M\}$.

(19) Let V be a real linear space, M be a non empty Affine subset of V, and v be a vector of V. If $v \in M$, then $M - \{v\} = \bigcup\{M - \{u\}; u$ ranges over vectors of V: $u \in M\}$.

(20) Let V be a real linear space and M be a non empty Affine subset of V. Then there exists a non empty Affine subset L of V such that $L = M - M$ and L is Subspace-like and M is parallel to L.

2. Orthogonality

Let V be a real unitary space and let W be a subspace of V. The functor Ort_Comp W yielding a strict subspace of V is defined by:

(Def. 3) The carrier of Ort_Comp W $=$ $\{v; v$ ranges over vectors of V: $\bigwedge_{w\,:\,\mathrm{vector\ of}\ V} (w \in W \Rightarrow w,\ v$ are orthogonal$)\}$.

Let V be a real unitary space and let M be a non empty subset of V. The functor Ort_Comp M yielding a strict subspace of V is defined by:

(Def. 4) The carrier of Ort_Comp M $=$ $\{v; v$ ranges over vectors of V: $\bigwedge_{w\,:\,\mathrm{vector\ of}\ V} (w \in M \Rightarrow w,\ v$ are orthogonal$)\}$.

One can prove the following propositions:

(21) For every real unitary space V and for every subspace W of V holds $0_V \in$ Ort_Comp W.

(22) For every real unitary space V holds Ort_Comp $\mathbf{0}_V = \Omega_V$.

(23) For every real unitary space V holds Ort_Comp $\Omega_V = \mathbf{0}_V$.

(24) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. If $v \neq 0_V$, then if $v \in W$, then $v \notin$ Ort_Comp W.

(25) For every real unitary space V and for every non empty subset M of V holds $M \subseteq$ the carrier of Ort_Comp Ort_Comp M.

(26) Let V be a real unitary space and M, N be non empty subsets of V. If $M \subseteq N$, then the carrier of Ort_Comp $N \subseteq$ the carrier of Ort_Comp M.

(27) Let V be a real unitary space, W be a subspace of V, and M be a non empty subset of V. If $M =$ the carrier of W, then Ort_Comp $M =$ Ort_Comp W.

(28) For every real unitary space V and for every non empty subset M of V holds Ort_Comp $M =$ Ort_Comp Ort_Comp Ort_Comp M.

(29) Let V be a real unitary space and x, y be vectors of V. Then $\|x+y\|^{\mathbf{2}} = \|x\|^{\mathbf{2}} + 2 \cdot (x|y) + \|y\|^{\mathbf{2}}$ and $\|x-y\|^{\mathbf{2}} = (\|x\|^{\mathbf{2}} - 2 \cdot (x|y)) + \|y\|^{\mathbf{2}}$.

(30) Let V be a real unitary space and x, y be vectors of V. If x, y are orthogonal, then $\|x+y\|^{\mathbf{2}} = \|x\|^{\mathbf{2}} + \|y\|^{\mathbf{2}}$.

(31) For every real unitary space V and for all vectors x, y of V holds $\|x+y\|^{\mathbf{2}} + \|x-y\|^{\mathbf{2}} = 2 \cdot \|x\|^{\mathbf{2}} + 2 \cdot \|y\|^{\mathbf{2}}$.

(32) Let V be a real unitary space and v be a vector of V. Then there exists a subspace W of V such that the carrier of $W = \{u; u$ ranges over vectors of V: $(u|v) = 0\}$.

3. Topology of Real Unitary Space

The scheme *SubFamExU* deals with a unitary space structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of the carrier of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

Let V be a real unitary space. The open set family of V yields a family of subsets of V and is defined by the condition (Def. 5).

(Def. 5) Let M be a subset of the carrier of V. Then $M \in$ the open set family of V if and only if for every point x of V such that $x \in M$ there exists a real number r such that $r > 0$ and $\mathrm{Ball}(x, r) \subseteq M$.

Next we state several propositions:

(33) Let V be a real unitary space, v be a point of V, and r, p be real numbers. If $r \leqslant p$, then $\mathrm{Ball}(v, r) \subseteq \mathrm{Ball}(v, p)$.

(34) Let V be a real unitary space and v be a point of V. Then there exists a real number r such that $r > 0$ and $\mathrm{Ball}(v, r) \subseteq$ the carrier of V.

(35) Let V be a real unitary space, v, u be points of V, and r be a real number. If $u \in \mathrm{Ball}(v, r)$, then there exists a real number p such that $p > 0$ and $\mathrm{Ball}(u, p) \subseteq \mathrm{Ball}(v, r)$.

(36) Let V be a real unitary space, u, v, w be points of V, and r, p be real numbers. If $v \in \mathrm{Ball}(u, r) \cap \mathrm{Ball}(w, p)$, then there exists a real number q such that $\mathrm{Ball}(v, q) \subseteq \mathrm{Ball}(u, r)$ and $\mathrm{Ball}(v, q) \subseteq \mathrm{Ball}(w, p)$.

(37) Let V be a real unitary space, v be a point of V, and r be a real number. Then $\mathrm{Ball}(v, r) \in$ the open set family of V.

(38) For every real unitary space V holds the carrier of $V \in$ the open set family of V.

(39) Let V be a real unitary space and M, N be subsets of the carrier of V. Suppose $M \in$ the open set family of V and $N \in$ the open set family of V. Then $M \cap N \in$ the open set family of V.

(40) Let V be a real unitary space and A be a family of subsets of the carrier of V. Suppose $A \subseteq$ the open set family of V. Then $\bigcup A \in$ the open set family of V.

(41) For every real unitary space V holds $\langle$the carrier of V, the open set family of $V\rangle$ is a topological space.

Let V be a real unitary space. The functor TopUnitSpace V yields a topological structure and is defined by:

(Def. 6) TopUnitSpace $V = \langle$the carrier of V, the open set family of $V\rangle$.

Let V be a real unitary space. Note that TopUnitSpace V is topological space-like.

Let V be a real unitary space. One can verify that TopUnitSpace V is non empty.

We now state a number of propositions:

(42) For every real unitary space V and for every subset M of TopUnitSpace V such that $M = \Omega_V$ holds M is open and closed.

(43) For every real unitary space V and for every subset M of TopUnitSpace V such that $M = \emptyset_V$ holds M is open and closed.

(44) Let V be a real unitary space, v be a vector of V, and r be a real number. If the carrier of $V = \{0_V\}$ and $r \neq 0$, then Sphere(v, r) is empty.

(45) Let V be a real unitary space, v be a vector of V, and r be a real number. If the carrier of $V \neq \{0_V\}$ and $r > 0$, then Sphere(v, r) is non empty.

(46) Let V be a real unitary space, v be a vector of V, and r be a real number. If $r = 0$, then Ball(v, r) is empty.

(47) Let V be a real unitary space, v be a vector of V, and r be a real number. If the carrier of $V = \{0_V\}$ and $r > 0$, then Ball$(v, r) = \{0_V\}$.

(48) Let V be a real unitary space, v be a vector of V, and r be a real number. Suppose the carrier of $V \neq \{0_V\}$ and $r > 0$. Then there exists a vector w of V such that $w \neq v$ and $w \in$ Ball(v, r).

(49) Let V be a real unitary space. Then the carrier of $V =$ the carrier of TopUnitSpace V and the topology of TopUnitSpace $V =$ the open set family of V.

(50) Let V be a real unitary space, M be a subset of TopUnitSpace V, r be a real number, and v be a point of V. If $M =$ Ball(v, r), then M is open.

(51) Let V be a real unitary space and M be a subset of TopUnitSpace V. Then M is open if and only if for every point v of V such that $v \in M$ there exists a real number r such that $r > 0$ and Ball$(v, r) \subseteq M$.

(52) Let V be a real unitary space, v_1, v_2 be points of V, and r_1, r_2 be real numbers. Then there exists a point u of V and there exists a real number r such that Ball$(v_1, r_1) \cup$ Ball$(v_2, r_2) \subseteq$ Ball(u, r).

(53) Let V be a real unitary space, M be a subset of TopUnitSpace V, v be a vector of V, and r be a real number. If $M = \overline{\mathrm{Ball}}(v, r)$, then M is closed.

(54) Let V be a real unitary space, M be a subset of TopUnitSpace V, v be a vector of V, and r be a real number. If $M = \mathrm{Sphere}(v, r)$, then M is closed.

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