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Quotient Vector Spaces and Functionals[1]

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Summary. The article presents well known facts about quotient vector spaces and functionals (see [8]). There are repeated theorems and constructions with either weaker assumptions or in more general situations (see [11], [7], [10]). The construction of coefficient functionals and non-degenerate functional in quotient vector space generated by functional in the given vector space are the only new things which are done.

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The articles [15], [5], [21], [13], [3], [1], [20], [2], [17], [7], [22], [4], [6], [14], [19], [12], [18], [16], and [9] provide the notation and terminology for this paper.

1. Auxiliary Facts about Double Loops and Vector Spaces

The following proposition is true

(1) Let K be an add-associative right zeroed right complementable left distributive left unital non empty double loop structure and a be an element of the carrier of K. Then $(-\mathbf{1}_K) \cdot a = -a$.

Let K be a double loop structure. The functor StructVectSp(K) yields a strict vector space structure over K and is defined as follows:

(Def. 1) StructVectSp(K) = $\langle$the carrier of K, the addition of K, the zero of K, the multiplication of $K\rangle$.

Let K be a non empty double loop structure. Note that StructVectSp(K) is non empty.

Let K be an Abelian non empty double loop structure. One can verify that StructVectSp(K) is Abelian.

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Let K be an add-associative non empty double loop structure. Note that StructVectSp(K) is add-associative.

Let K be a right zeroed non empty double loop structure.

Note that StructVectSp(K) is right zeroed.

Let K be a right complementable non empty double loop structure. Observe that StructVectSp(K) is right complementable.

Let K be an associative left unital distributive non empty double loop structure. One can check that StructVectSp(K) is vector space-like.

Let K be a non degenerated non empty double loop structure. Note that StructVectSp(K) is non trivial.

Let K be a non degenerated non empty double loop structure. Note that there exists a non empty vector space structure over K which is non trivial.

Let K be an add-associative right zeroed right complementable non empty double loop structure. Observe that there exists a non empty vector space structure over K which is add-associative, right zeroed, right complementable, and strict.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure. One can check that there exists a non empty vector space structure over K which is add-associative, right zeroed, right complementable, vector space-like, and strict.

Let K be an Abelian add-associative right zeroed right complementable associative left unital distributive non degenerated non empty double loop structure. One can verify that there exists a non empty vector space structure over K which is Abelian, add-associative, right zeroed, right complementable, vector space-like, strict, and non trivial.

Next we state a number of propositions:

(2) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, a be an element of the carrier of K, V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and v be a vector of V. Then $0_K \cdot v = 0_V$ and $a \cdot 0_V = 0_V$.

(3) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, S, T be subspaces of V, and v be a vector of V. If $S \cap T = \mathbf{0}_V$ and $v \in S$ and $v \in T$, then $v = 0_V$.

(4) Let K be a field, V be a vector space over K, x be a set, and v be a vector of V. Then $x \in \mathrm{Lin}(\{v\})$ if and only if there exists an element a of the carrier of K such that $x = a \cdot v$.

(5) Let K be a field, V be a vector space over K, v be a vector of V, and a, b be scalars of V. If $v \neq 0_V$ and $a \cdot v = b \cdot v$, then $a = b$.

(6) Let K be an add-associative right zeroed right complementable associa-

tive Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$, then $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.

(7) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v = v_1 + v_2$.

(8) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_1 \in W_1$ and $v_2 \in W_2$.

(9) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_{\langle W_2, W_1 \rangle} = \langle v_2, v_1 \rangle$.

(10) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_1$, then $v_{\langle W_1, W_2 \rangle} = \langle v, 0_V \rangle$.

(11) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_2$, then $v_{\langle W_1, W_2 \rangle} = \langle 0_V, v \rangle$.

(12) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, V_1 be a subspace of V, W_1 be a subspace of V_1, and v be a vector of V. If $v \in W_1$, then v is a vector of V_1.

(13) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, V_1, V_2, W be subspaces of V, and W_1, W_2 be subspaces of W. If $W_1 = V_1$ and $W_2 = V_2$, then $W_1 + W_2 = V_1 + V_2$.

(14) Let K be a field, V be a vector space over K, W be a subspace of V, v be a vector of V, and w be a vector of W. If $v = w$, then $\mathrm{Lin}(\{w\}) = \mathrm{Lin}(\{v\})$.

(15) Let K be a field, V be a vector space over K, v be a vector of V, and X be a subspace of V. Suppose $v \notin X$. Let y be a vector of $X + \mathrm{Lin}(\{v\})$ and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $W = X$, then $X + \mathrm{Lin}(\{v\})$ is the direct sum of W and $\mathrm{Lin}(\{y\})$.

(16) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $X = W$ and $v \notin X$, then $y_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle 0_W, y \rangle$.

(17) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. If $w \in X$, then $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle w, 0_V \rangle$.

(18) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, v be a vector of V, and W_1, W_2 be subspaces of V. Then there exist vectors v_1, v_2 of V such that $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.

(19) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. Then there exists a vector x of X and there exists an element r of the carrier of K such that $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x, r \cdot v \rangle$.

(20) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w_1, w_2 be vectors of $X + \mathrm{Lin}(\{v\})$, x_1, x_2 be vectors of X, and r_1, r_2 be elements of the carrier of K. If $(w_1)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x_1, r_1 \cdot v \rangle$ and $(w_2)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x_2, r_2 \cdot v \rangle$, then $(w_1 + w_2)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x_1 + x_2, (r_1 + r_2) \cdot v \rangle$.

(21) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$, x be a vector of X, and t, r be elements of the carrier of K. If $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x, r \cdot v \rangle$, then $(t \cdot w)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle t \cdot x, t \cdot r \cdot v \rangle$.

2. Quotient Vector Space for Non-Commutative Double Loop

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor $\mathrm{CosetSet}(V, W)$ yielding a non empty family of subsets of the carrier of V is defined as follows:

(Def. 2) CosetSet$(V, W) = \{A : A$ ranges over cosets of $W\}$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor addCoset(V, W) yields a binary operation on CosetSet(V, W) and is defined by:

(Def. 3) For all elements A, B of CosetSet(V, W) and for all vectors a, b of V such that $A = a+W$ and $B = b+W$ holds (addCoset$(V, W))(A, B) = a+b+W$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor zeroCoset(V, W) yielding an element of CosetSet(V, W) is defined as follows:

(Def. 4) zeroCoset$(V, W) =$ the carrier of W.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor lmultCoset(V, W) yields a function from [: the carrier of K, CosetSet(V, W) :] into CosetSet(V, W) and is defined by the condition (Def. 5).

(Def. 5) Let z be an element of the carrier of K, A be an element of CosetSet(V, W), and a be a vector of V. If $A = a + W$, then (lmultCoset$(V, W))(z, A) = z \cdot a + W$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor ${}^{V}/_{W}$ yielding a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over K is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of ${}^{V}/_{W} =$ CosetSet(V, W),

(ii) the addition of ${}^{V}/_{W} =$ addCoset(V, W),

(iii) the zero of ${}^{V}/_{W} =$ zeroCoset(V, W), and

(iv) the left multiplication of ${}^{V}/_{W} =$ lmultCoset(V, W).

The following propositions are true:

(22) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, and W be a subspace of V. Then zeroCoset$(V, W) = 0_V + W$ and $0_{{}^{V}/_{W}} =$ zeroCoset(V, W).

(23) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, and w be a vector of ${}^{V}/_{W}$. Then w is a coset of W and there exists a vector v of V such that $w = v + W$.

(24) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, and v be a vector of V. Then $v + W$ is a coset of W and $v + W$ is a vector of ${}^{V}/_{W}$.

(25) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, and W be a subspace of V. Then every coset of W is a vector of ${}^{V}/_{W}$.

(26) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, A be a vector of ${}^{V}/_{W}$, v be a vector of V, and a be a scalar of V. If $A = v + W$, then $a \cdot A = a \cdot v + W$.

(27) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, A_1, A_2 be vectors of ${}^{V}/_{W}$, and v_1, v_2 be vectors of V. If $A_1 = v_1 + W$ and $A_2 = v_2 + W$, then $A_1 + A_2 = v_1 + v_2 + W$.

3. Auxiliary Facts about Functionals

Next we state the proposition

(28) Let K be a field, V be a vector space over K, X be a subspace of V, f_1 be a linear functional in X, v be a vector of V, and y be a vector of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $v \notin X$. Let r be an element of the carrier of K. Then there exists a linear functional p_1 in $X + \mathrm{Lin}(\{v\})$ such that $p_1 \restriction$the carrier of $X = f_1$ and $p_1(y) = r$.

Let K be a right zeroed non empty loop structure and let V be a non empty vector space structure over K. One can verify that there exists a functional in V which is additive and 0-preserving.

Let K be an add-associative right zeroed right complementable non empty double loop structure and let V be a right zeroed non empty vector space structure over K. Observe that every functional in V which is additive is also 0-preserving.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure and let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K. One can verify that every functional in V which is homogeneous is also 0-preserving.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. One can check that 0Functional V is constant.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. Note that there exists a functional in V which is constant.

Let K be an add-associative right zeroed right complementable non empty double loop structure, let V be a right zeroed non empty vector space structure over K, and let f be a 0-preserving functional in V. Let us observe that f is constant if and only if:

(Def. 7) $f = 0\mathrm{Functional}\,V$.

Let K be an add-associative right zeroed right complementable non empty double loop structure and let V be a right zeroed non empty vector space structure over K. Note that there exists a functional in V which is constant, additive, and 0-preserving.

Let K be a non empty 1-sorted structure and let V be a non empty vector space structure over K. One can check that every functional in V which is non constant is also non trivial.

Let K be a field and let V be a non trivial vector space over K. Observe that there exists a functional in V which is additive, homogeneous, non constant, and non trivial.

Let K be a field and let V be a non trivial vector space over K. One can check that every functional in V which is trivial is also constant.

Let K be a field, let V be a non trivial vector space over K, let v be a vector of V, and let W be a linear complement of $\mathrm{Lin}(\{v\})$. Let us assume that $v \neq 0_V$. The functor $\mathrm{coeffFunctional}(v, W)$ yielding a non constant non trivial linear functional in V is defined as follows:

(Def. 8) $(\mathrm{coeffFunctional}(v, W))(v) = \mathbf{1}_K$ and $\mathrm{coeffFunctional}(v, W) \restriction$ the carrier of $W = 0\mathrm{Functional}\,W$.

We now state several propositions:

(29) Let K be a field, V be a non trivial vector space over K, and f be a non constant 0-preserving functional in V. Then there exists a vector v of V such that $v \neq 0_V$ and $f(v) \neq 0_K$.

(30) Let K be a field, V be a non trivial vector space over K, v be a vector of V, a be a scalar of V, and W be a linear complement of $\mathrm{Lin}(\{v\})$. If $v \neq 0_V$, then $(\mathrm{coeffFunctional}(v, W))(a \cdot v) = a$.

(31) Let K be a field, V be a non trivial vector space over K, v, w be vectors of V, and W be a linear complement of $\mathrm{Lin}(\{v\})$. If $v \neq 0_V$ and $w \in W$, then $(\mathrm{coeffFunctional}(v, W))(w) = 0_K$.

(32) Let K be a field, V be a non trivial vector space over K, v, w be vectors of V, a be a scalar of V, and W be a linear complement of $\mathrm{Lin}(\{v\})$. If $v \neq 0_V$ and $w \in W$, then $(\mathrm{coeffFunctional}(v, W))(a \cdot v + w) = a$.

(33) Let K be a non empty loop structure, V be a non empty vector space structure over K, f, g be functionals in V, and v be a vector of V. Then

$(f - g)(v) = f(v) - g(v)$.

Let K be a field and let V be a non trivial vector space over K. Note that $\overline{V}$ is non trivial.

4. Kernel of Additive Functional. Linear Functionals in Quotient Vector Spaces

Let K be a non empty zero structure, let V be a non empty vector space structure over K, and let f be a functional in V. The functor $\ker f$ yields a subset of the carrier of V and is defined by:

(Def. 9) $\ker f = \{v; v$ ranges over vectors of V: $f(v) = 0_K\}$.

Let K be a right zeroed non empty loop structure, let V be a non empty vector space structure over K, and let f be a 0-preserving functional in V. One can check that $\ker f$ is non empty.

One can prove the following proposition

(34) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and f be a linear functional in V. Then $\ker f$ is linearly closed.

Let K be a non empty zero structure, let V be a non empty vector space structure over K, and let f be a functional in V. We say that f is degenerated if and only if:

(Def. 10) $\ker f \neq \{0_V\}$.

Let K be a non degenerated non empty double loop structure and let V be a non trivial non empty vector space structure over K. One can check that every functional in V which is non degenerated and 0-preserving is also non constant.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let f be a linear functional in V. The functor $\mathrm{Ker}\, f$ yields a strict non empty subspace of V and is defined as follows:

(Def. 11) The carrier of $\mathrm{Ker}\, f = \ker f$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, let W be a subspace of V, and let f be an additive functional in V. Let us assume that the carrier of $W \subseteq \ker f$. The functor ${}^{f}/_{W}$ yielding an additive functional in ${}^{V}/_{W}$ is defined by:

(Def. 12) For every vector A of ${}^{V}/_{W}$ and for every vector v of V such that $A = v + W$ holds $({}^{f}/_{W})(A) = f(v)$.

One can prove the following proposition

(35) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, and f be a linear functional in V. If the carrier of $W \subseteq \ker f$, then ${}^{f}/_{W}$ is homogeneous.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let f be a linear functional in V. The functor CQFunctional f yielding a linear functional in ${}^{V}/_{\mathrm{Ker}\, f}$ is defined as follows:

(Def. 13) $\mathrm{CQFunctional}\, f = {}^{f}/_{\mathrm{Ker}\, f}$.

One can prove the following proposition

(36) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, f be a linear functional in V, A be a vector of ${}^{V}/_{\mathrm{Ker}\, f}$, and v be a vector of V. If $A = v + \mathrm{Ker}\, f$, then $\mathrm{CQFunctional}\, f(A) = f(v)$.

Let K be a field, let V be a non trivial vector space over K, and let f be a non constant linear functional in V. Observe that CQFunctional f is non constant.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let f be a linear functional in V. One can verify that CQFunctional f is non degenerated.

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