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Bessel's Inequality

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Summary. In this article we defined the operation of a set and proved Bessel's inequality. In the first section, we defined the sum of all results of an operation, in which the results are given by taking each element of a set. In the second section, we defined Orthogonal Family and Orthonormal Family. In the last section, we proved some properties of operation of set and Bessel's inequality.

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The articles [12], [16], [10], [7], [5], [6], [17], [15], [9], [13], [3], [8], [1], [11], [4], [2], and [14] provide the terminology and notation for this paper.

1. Sum of the Result of Operation with Each Element of a Set

For simplicity, we adopt the following convention: X denotes a real unitary space, x, y, y_1, y_2 denote points of X, i, j denote natural numbers, D_1 denotes a non empty set, and p_1, p_2 denote finite sequences of elements of D_1.

Next we state the proposition

(1) Suppose p_1 is one-to-one and p_2 is one-to-one and $\operatorname{rng} p_1 = \operatorname{rng} p_2$. Then $\operatorname{dom} p_1 = \operatorname{dom} p_2$ and there exists a permutation P of $\operatorname{dom} p_1$ such that $p_2 = p_1 \cdot P$ and $\operatorname{dom} P = \operatorname{dom} p_1$ and $\operatorname{rng} P = \operatorname{dom} p_1$.

Let D_1 be a non empty set and let f be a binary operation on D_1. Let us assume that f is commutative and associative and has a unity. Let Y be a finite subset of D_1. The functor $f \oplus Y$ yields an element of D_1 and is defined as follows:

(Def. 1) There exists a finite sequence p of elements of D_1 such that p is one-to-one and $\operatorname{rng} p = Y$ and $f \oplus Y = f \odot p$.

Let us consider X and let Y be a finite subset of the carrier of X. The functor $\operatorname{SetopSum}(Y, X)$ is defined as follows:

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(Def. 2) $\mathrm{SetopSum}(Y, X) = \begin{cases} (\text{the addition of } X) \oplus Y, & \text{if } Y \neq \emptyset, \\ 0_X, & \text{otherwise.} \end{cases}$

Let us consider X, x, let p be a finite sequence, and let us consider i. The functor $\mathrm{PO}(i, p, x)$ is defined by:

(Def. 3) $\mathrm{PO}(i, p, x) = (\text{the scalar product of } X)(\langle x, p(i) \rangle)$.

Let D_2, D_1 be non empty sets, let F be a function from D_1 into D_2, and let p be a finite sequence of elements of D_1. The functor $\mathrm{FuncSeq}(F, p)$ yielding a finite sequence of elements of D_2 is defined as follows:

(Def. 4) $\mathrm{FuncSeq}(F, p) = F \cdot p$.

Let D_2, D_1 be non empty sets and let f be a binary operation on D_2. Let us assume that f is commutative and associative and has a unity. Let Y be a finite subset of D_1 and let F be a function from D_1 into D_2. Let us assume that $Y \subseteq \mathrm{dom}\, F$. The functor $\mathrm{setopfunc}(Y, D_1, D_2, F, f)$ yielding an element of D_2 is defined by:

(Def. 5) There exists a finite sequence p of elements of D_1 such that p is one-to-one and $\mathrm{rng}\, p = Y$ and $\mathrm{setopfunc}(Y, D_1, D_2, F, f) = f \odot \mathrm{FuncSeq}(F, p)$.

Let us consider X, x and let Y be a finite subset of the carrier of X. The functor $\mathrm{SetopPreProd}(x, Y, X)$ yields a real number and is defined by the condition (Def. 6).

(Def. 6) There exists a finite sequence p of elements of the carrier of X such that

(i) p is one-to-one,

(ii) $\mathrm{rng}\, p = Y$, and

(iii) there exists a finite sequence q of elements of $\mathbb{R}$ such that $\mathrm{dom}\, q = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, q$ holds $q(i) = \mathrm{PO}(i, p, x)$ and $\mathrm{SetopPreProd}(x, Y, X) = +_{\mathbb{R}} \odot q$.

Let us consider X, x and let Y be a finite subset of the carrier of X. The functor $\mathrm{SetopProd}(x, Y, X)$ yielding a real number is defined as follows:

(Def. 7) $\mathrm{SetopProd}(x, Y, X) = \begin{cases} \mathrm{SetopPreProd}(x, Y, X), & \text{if } Y \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$

2. Orthogonal Family and Orthonormal Family

Let us consider X. A subset of the carrier of X is said to be an orthogonal family of X if:

(Def. 8) For all x, y such that $x \in$ it and $y \in$ it and $x \neq y$ holds $(x|y) = 0$.

The following proposition is true

(2) $\emptyset$ is an orthogonal family of X.

Let us consider X. Observe that there exists an orthogonal family of X which is finite.

Let us consider X. A subset of the carrier of X is said to be an orthonormal family of X if:

(Def. 9) It is an orthogonal family of X and for every x such that $x \in$ it holds $(x|x) = 1$.

One can prove the following proposition

(3) $\emptyset$ is an orthonormal family of X.

Let us consider X. One can check that there exists an orthonormal family of X which is finite.

The following proposition is true

(4) $x = 0_X$ iff for every y holds $(x|y) = 0$.

3. Bessel's Inequality

We now state a number of propositions:

(5) $\|x + y\|^{\mathbf{2}} + \|x - y\|^{\mathbf{2}} = 2 \cdot \|x\|^{\mathbf{2}} + 2 \cdot \|y\|^{\mathbf{2}}$.

(6) If x, y are orthogonal, then $\|x + y\|^{\mathbf{2}} = \|x\|^{\mathbf{2}} + \|y\|^{\mathbf{2}}$.

(7) Let p be a finite sequence of elements of the carrier of X. Suppose $\operatorname{len} p \geqslant 1$ and for all i, j such that $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $i \neq j$ holds (the scalar product of X)($\langle p(i), p(j) \rangle$) $= 0$. Let q be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{dom} p = \operatorname{dom} q$ and for every i such that $i \in \operatorname{dom} q$ holds $q(i) =$ (the scalar product of X)($\langle p(i), p(i) \rangle$). Then ((the addition of $X \odot p$)|(the addition of $X \odot p$)) $= +_{\mathbb{R}} \odot q$.

(8) Let p be a finite sequence of elements of the carrier of X. Suppose $\operatorname{len} p \geqslant 1$. Let q be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{dom} p = \operatorname{dom} q$ and for every i such that $i \in \operatorname{dom} q$ holds $q(i) =$ (the scalar product of X)($\langle x, p(i) \rangle$). Then (x|(the addition of $X \odot p$)) $= +_{\mathbb{R}} \odot q$.

(9) Let S be a finite non empty subset of the carrier of X and F be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \operatorname{dom} F$ and for all y_1, y_2 such that $y_1 \in S$ and $y_2 \in S$ and $y_1 \neq y_2$ holds (the scalar product of X)($\langle F(y_1), F(y_2) \rangle$) $= 0$. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) =$ (the scalar product of X)($\langle F(y), F(y) \rangle$). Let p be a finite sequence of elements of the carrier of X. Suppose p is one-to-one and $\operatorname{rng} p = S$. Then (the scalar product of X)($\langle$the addition of $X \odot \operatorname{FuncSeq}(F, p)$, the addition of $X \odot \operatorname{FuncSeq}(F, p)\rangle$) $= +_{\mathbb{R}} \odot \operatorname{FuncSeq}(H, p)$.

(10) Let S be a finite non empty subset of the carrier of X and F be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \operatorname{dom} F$. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) =$ (the scalar product of X)($\langle x, F(y) \rangle$). Let

p be a finite sequence of elements of the carrier of X. Suppose p is one-to-one and $\operatorname{rng} p = S$. Then (the scalar product of X)($\langle x$, the addition of $X \odot \operatorname{FuncSeq}(F, p)\rangle$) $= +_{\mathbb{R}} \odot \operatorname{FuncSeq}(H, p)$.

(11) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let given x and S be a finite orthonormal family of X. Suppose S is non empty. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) = (x|y)^{\mathbf{2}}$. Let F be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \operatorname{dom} F$ and for every y such that $y \in S$ holds $F(y) = (x|y) \cdot y$. Then $(x|$ setopfunc(S, the carrier of X, the carrier of X, F, the addition of X)) = setopfunc(S, the carrier of X, $\mathbb{R}, H, +_{\mathbb{R}}$).

(12) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let given x and S be a finite orthonormal family of X. Suppose S is non empty. Let F be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \operatorname{dom} F$ and for every y such that $y \in S$ holds $F(y) = (x|y) \cdot y$. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) = (x|y)^{\mathbf{2}}$. Then (setopfunc($S$, the carrier of X, the carrier of X, F, the addition of X)| setopfunc(S, the carrier of X, the carrier of X, F, the addition of X)) = setopfunc(S, the carrier of X, $\mathbb{R}, H, +_{\mathbb{R}}$).

(13) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let given x and S be a finite orthonormal family of X. Suppose S is non empty. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \operatorname{dom} H$ and for every y such that $y \in S$ holds $H(y) = (x|y)^{\mathbf{2}}$. Then setopfunc($S$, the carrier of X, $\mathbb{R}, H, +_{\mathbb{R}}) \leqslant \|x\|^{\mathbf{2}}$.

(14) Let D_2, D_1 be non empty sets and f be a binary operation on D_2. Suppose f is commutative and associative and has a unity. Let Y_1, Y_2 be finite subsets of D_1. Suppose Y_1 misses Y_2. Let F be a function from D_1 into D_2. Suppose $Y_1 \subseteq \operatorname{dom} F$ and $Y_2 \subseteq \operatorname{dom} F$. Let Z be a finite subset of D_1. If $Z = Y_1 \cup Y_2$, then setopfunc$(Z, D_1, D_2, F, f) = f($setopfunc(Y_1, D_1, D_2, F, f), setopfunc$(Y_2, D_1, D_2, F, f))$.

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