

# A Representation of Integers by Binary Arithmetics and Addition of Integers

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**Summary.** In this article, we introduce the new concept of 2's complement representation. Natural numbers that are congruent mod  $n$  can be represented by the same  $n$  bits binary. Using the concept introduced here, negative numbers that are congruent mod  $n$  also can be represented by the same  $n$  bit binary. We also show some properties of addition of integers using this concept.

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The articles [16], [20], [2], [3], [12], [11], [10], [9], [17], [13], [14], [6], [7], [1], [15], [18], [4], [21], [8], [5], and [19] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

We follow the rules:  $n$  denotes a non empty natural number,  $j, k, l, m$  denote natural numbers, and  $g, h, i$  denote integers.

We now state a number of propositions:

- (1) If  $m > 0$ , then  $m \cdot 2 \geq m + 1$ .
- (2) For every natural number  $m$  holds  $2^m \geq m$ .
- (3) For every natural number  $m$  holds  $\underbrace{\langle 0, \dots, 0 \rangle}_m + \underbrace{\langle 0, \dots, 0 \rangle}_m = \underbrace{\langle 0, \dots, 0 \rangle}_m$ .
- (4) For every natural number  $k$  such that  $k \leq l$  and  $l \leq m$  holds  $k = l$  or  $k + 1 \leq l$  and  $l \leq m$ .

- (5) For every non empty natural number  $n$  and for all  $n$ -tuples  $x, y$  of *Boolean* such that  $x = \underbrace{\langle 0, \dots, 0 \rangle}_n$  and  $y = \underbrace{\langle 0, \dots, 0 \rangle}_n$  holds  $\text{carry}(x, y) = \underbrace{\langle 0, \dots, 0 \rangle}_n$ .
- (6) For every non empty natural number  $n$  and for all  $n$ -tuples  $x, y$  of *Boolean* such that  $x = \underbrace{\langle 0, \dots, 0 \rangle}_n$  and  $y = \underbrace{\langle 0, \dots, 0 \rangle}_n$  holds  $x+y = \underbrace{\langle 0, \dots, 0 \rangle}_n$ .
- (7) For every non empty natural number  $n$  and for every  $n$ -tuple  $F$  of *Boolean* such that  $F = \underbrace{\langle 0, \dots, 0 \rangle}_n$  holds  $\text{Intval}(F) = 0$ .
- (8) If  $l + m \leq k - 1$ , then  $l < k$  and  $m < k$ .
- (9) If  $g \leq h + i$  and  $h < 0$  and  $i < 0$ , then  $g < h$  and  $g < i$ .
- (10) If  $l + m \leq 2^n - 1$ , then  $\text{add\_ovfl}(n\text{-BinarySequence}(l), n\text{-BinarySequence}(m)) = \text{false}$ .
- (11) For every non empty natural number  $n$  and for all natural numbers  $l, m$  such that  $l + m \leq 2^n - 1$  holds  $\text{Absval}((n\text{-BinarySequence}(l)) + (n\text{-BinarySequence}(m))) = l + m$ .
- (12) For every non empty natural number  $n$  and for every  $n$ -tuple  $z$  of *Boolean* such that  $z_n = \text{true}$  holds  $\text{Absval}(z) \geq 2^{n-1}$ .
- (13) If  $l + m \leq 2^{n-1} - 1$ , then  $(\text{carry}(n\text{-BinarySequence}(l), n\text{-BinarySequence}(m)))_n = \text{false}$ .
- (14) For every non empty natural number  $n$  such that  $l + m \leq 2^{n-1} - 1$  holds  $\text{Intval}((n\text{-BinarySequence}(l)) + (n\text{-BinarySequence}(m))) = l + m$ .
- (15) For every 1-tuple  $z$  of *Boolean* such that  $z = \langle \text{true} \rangle$  holds  $\text{Intval}(z) = -1$ .
- (16) For every 1-tuple  $z$  of *Boolean* such that  $z = \langle \text{false} \rangle$  holds  $\text{Intval}(z) = 0$ .
- (17) For every boolean set  $x$  holds  $\text{true} \vee x = \text{true}$ .
- (18) For every non empty natural number  $n$  holds  $0 \leq 2^{n-1} - 1$  and  $-2^{n-1} \leq 0$ .
- (19) For all  $n$ -tuples  $x, y$  of *Boolean* such that  $x = \underbrace{\langle 0, \dots, 0 \rangle}_n$  and  $y = \underbrace{\langle 0, \dots, 0 \rangle}_n$  holds  $x$  and  $y$  are summable.
- (20)  $i \cdot n \bmod n = 0$ .

## 2. MAJORANT POWER

Let  $m, j$  be natural numbers. The functor  $\text{MajP}(m, j)$  yielding a natural number is defined as follows:

(Def. 1)  $2^{\text{MajP}(m,j)} \geq j$  and  $\text{MajP}(m,j) \geq m$  and for every natural number  $k$  such that  $2^k \geq j$  and  $k \geq m$  holds  $k \geq \text{MajP}(m,j)$ .

One can prove the following propositions:

- (21) If  $j \geq k$ , then  $\text{MajP}(m,j) \geq \text{MajP}(m,k)$ .
- (22) If  $l \geq m$ , then  $\text{MajP}(l,j) \geq \text{MajP}(m,j)$ .
- (23) If  $m \geq 1$ , then  $\text{MajP}(m,1) = m$ .
- (24) If  $j \leq 2^m$ , then  $\text{MajP}(m,j) = m$ .
- (25) If  $j > 2^m$ , then  $\text{MajP}(m,j) > m$ .

### 3. 2'S COMPLEMENT

Let  $m$  be a natural number and let  $i$  be an integer.

The functor  $2\text{sComplement}(m,i)$  yields a  $m$ -tuple of *Boolean* and is defined by:

(Def. 2)  $2\text{sComplement}(m,i) = \begin{cases} m\text{-BinarySequence}(|2^{\text{MajP}(m,|i|)} + i|), & \text{if } i < 0, \\ m\text{-BinarySequence}(|i|), & \text{otherwise.} \end{cases}$

The following propositions are true:

- (26) For every natural number  $m$  holds  $2\text{sComplement}(m,0) = \underbrace{\langle 0, \dots, 0 \rangle}_m$ .
- (27) For every integer  $i$  such that  $i \leq 2^{n-1} - 1$  and  $-2^{n-1} \leq i$  holds  $\text{Intval}(2\text{sComplement}(n,i)) = i$ .
- (28) For all integers  $h, i$  such that  $h \geq 0$  and  $i \geq 0$  or  $h < 0$  and  $i < 0$  but  $h \bmod 2^n = i \bmod 2^n$  holds  $2\text{sComplement}(n,h) = 2\text{sComplement}(n,i)$ .
- (29) For all integers  $h, i$  such that  $h \geq 0$  and  $i \geq 0$  or  $h < 0$  and  $i < 0$  but  $h \equiv i \pmod{2^n}$  holds  $2\text{sComplement}(n,h) = 2\text{sComplement}(n,i)$ .
- (30) For all natural numbers  $l, m$  such that  $l \bmod 2^n = m \bmod 2^n$  holds  $n\text{-BinarySequence}(l) = n\text{-BinarySequence}(m)$ .
- (31) For all natural numbers  $l, m$  such that  $l \equiv m \pmod{2^n}$  holds  $n\text{-BinarySequence}(l) = n\text{-BinarySequence}(m)$ .
- (32) For every natural number  $j$  such that  $1 \leq j$  and  $j \leq n$  holds  $(2\text{sComplement}(n+1,i))_j = (2\text{sComplement}(n,i))_j$ .
- (33) There exists an element  $x$  of *Boolean* such that  $2\text{sComplement}(m+1,i) = (2\text{sComplement}(m,i)) \wedge \langle x \rangle$ .
- (34) There exists an element  $x$  of *Boolean* such that  $(m+1)\text{-BinarySequence}(l) = (m\text{-BinarySequence}(l)) \wedge \langle x \rangle$ .
- (35) Let  $n$  be a non empty natural number. Suppose  $-2^n \leq h + i$  and  $h < 0$  and  $i < 0$  and  $-2^{n-1} \leq h$  and  $-2^{n-1} \leq i$ . Then  $(\text{carry}(2\text{sComplement}(n+1,h), 2\text{sComplement}(n+1,i)))_{n+1} = \text{true}$ .

- (36) For every non empty natural number  $n$  such that  $-2^{n-1} \leq h + i$  and  $h + i \leq 2^{n-1} - 1$  and  $h \geq 0$  and  $i \geq 0$  holds  $\text{Intval}(2\text{sComplement}(n, h) + 2\text{sComplement}(n, i)) = h + i$ .
- (37) Let  $n$  be a non empty natural number. Suppose  $-2^{(n+1)-1} \leq h + i$  and  $h + i \leq 2^{(n+1)-1} - 1$  and  $h < 0$  and  $i < 0$  and  $-2^{n-1} \leq h$  and  $-2^{n-1} \leq i$ . Then  $\text{Intval}(2\text{sComplement}(n + 1, h) + 2\text{sComplement}(n + 1, i)) = h + i$ .
- (38) Let  $n$  be a non empty natural number. Suppose that  $-2^{n-1} \leq h$  and  $h \leq 2^{n-1} - 1$  and  $-2^{n-1} \leq i$  and  $i \leq 2^{n-1} - 1$  and  $-2^{n-1} \leq h + i$  and  $h + i \leq 2^{n-1} - 1$  and  $h \geq 0$  and  $i < 0$  or  $h < 0$  and  $i \geq 0$ . Then  $\text{Intval}(2\text{sComplement}(n, h) + 2\text{sComplement}(n, i)) = h + i$ .

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