FORMALIZED MATHEMATICS
Volume 11, Number 2, 2003
University of Białystok

Chains on a Grating in Euclidean Space[1]

Freek Wiedijk
University of Nijmegen

Summary. Translation of pages 101, the second half of 102, and 103 of [15].

MML Identifier: `CHAIN_1`.

The notation and terminology used here are introduced in the following papers: [20], [10], [22], [23], [18], [8], [12], [9], [17], [1], [19], [14], [3], [6], [13], [16], [2], [11], [4], [7], [21], and [5].

1. Preliminaries

We use the following convention: X, x, y, z are sets and n, m, k, k', d' are natural numbers.

The following two propositions are true:

(1) For all real numbers x, y such that $x < y$ there exists a real number z such that $x < z$ and $z < y$.

(2) For all real numbers x, y there exists a real number z such that $x < z$ and $y < z$.

The scheme *FrSet 1 2* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

$$\{\mathcal{F}(x,y); x \text{ ranges over elements of } \mathcal{B}, y \text{ ranges over elements of } \mathcal{B} : \mathcal{P}[x,y]\} \subseteq \mathcal{A}$$

for all values of the parameters.

Let B be a set and let A be a subset of B. Then 2^A is a subset of 2^B.

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102 and TYPES grant IST-1999-29001.

ISSN 1426–2630

Let X be a set. A subset of X is an element of 2^X.

Let d be a real natural number. Let us observe that d is zero if and only if:

(Def. 1) $d \not> 0$.

Let d be a natural number. Let us observe that d is zero if and only if:

(Def. 2) $d \not\geq 1$.

Let us note that there exists a natural number which is non zero.

In the sequel d denotes a non zero natural number.

Let us consider d. Observe that $\mathrm{Seg}\,d$ is non empty.

In the sequel i, i_0 denote elements of $\mathrm{Seg}\,d$.

Let us consider X. Let us observe that X is trivial if and only if:

(Def. 3) For all x, y such that $x \in X$ and $y \in X$ holds $x = y$.

Next we state the proposition

(4)[2] $\{x, y\}$ is trivial iff $x = y$.

Let us observe that there exists a set which is non trivial and finite.

Let X be a non trivial set and let Y be a set. Note that $X \cup Y$ is non trivial and $Y \cup X$ is non trivial.

Let us observe that $\mathbb{R}$ is non trivial.

Let X be a non trivial set. Observe that there exists a subset of X which is non trivial and finite.

The following proposition is true

(5) If X is trivial and $X \cup \{y\}$ is non trivial, then there exists x such that $X = \{x\}$.

Now we present two schemes. The scheme *NonEmptyFinite* deals with a non empty set $\mathcal{A}$, a non empty finite subset $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following requirements are met:

- For every element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ holds $\mathcal{P}[\{x\}]$, and
- Let x be an element of $\mathcal{A}$ and B be a non empty finite subset of $\mathcal{A}$. If $x \in \mathcal{B}$ and $B \subseteq \mathcal{B}$ and $x \notin B$ and $\mathcal{P}[B]$, then $\mathcal{P}[B \cup \{x\}]$.

The scheme *NonTrivialFinite* deals with a non trivial set $\mathcal{A}$, a non trivial finite subset $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the following conditions are met:

- For all elements x, y of $\mathcal{A}$ such that $x \in \mathcal{B}$ and $y \in \mathcal{B}$ and $x \neq y$ holds $\mathcal{P}[\{x, y\}]$, and
- Let x be an element of $\mathcal{A}$ and B be a non trivial finite subset of $\mathcal{A}$. If $x \in \mathcal{B}$ and $B \subseteq \mathcal{B}$ and $x \notin B$ and $\mathcal{P}[B]$, then $\mathcal{P}[B \cup \{x\}]$.

Next we state the proposition

[2]The proposition (3) has been removed.

(6) $\overline{\overline{X}} = 2$ iff there exist x, y such that $x \in X$ and $y \in X$ and $x \neq y$ and for every z such that $z \in X$ holds $z = x$ or $z = y$.

Let X, Y be finite sets. Note that $X \dot{-} Y$ is finite.

We now state three propositions:

(7) m is even iff n is even iff $m + n$ is even.

(8) Let X, Y be finite sets. Suppose X misses Y. Then $\operatorname{card} X$ is even iff $\operatorname{card} Y$ is even if and only if $\operatorname{card}(X \cup Y)$ is even.

(9) For all finite sets X, Y holds $\operatorname{card} X$ is even iff $\operatorname{card} Y$ is even iff $\operatorname{card}(X \dot{-} Y)$ is even.

Let us consider n. Then $\mathcal{R}^n$ can be characterized by the condition:

(Def. 4) For every x holds $x \in \mathcal{R}^n$ iff x is a function from $\operatorname{Seg} n$ into $\mathbb{R}$.

We adopt the following rules: l, r, l', r', x are elements of $\mathcal{R}^d$, G_1 is a non trivial finite subset of $\mathbb{R}$, and l_1, r_1, l'_1, r'_1, x_1 are real numbers.

Let us consider d, x, i. Then $x(i)$ is a real number.

2. Gratings, Cells, Chains, Cycles

Let us consider d. A function from $\operatorname{Seg} d$ into $2^{\mathbb{R}}$ is said to be a d-dimensional grating if:

(Def. 5) For every i holds $\mathrm{it}(i)$ is non trivial and finite.

In the sequel G is a d-dimensional grating.

Let us consider d, G, i. Then $G(i)$ is a non trivial finite subset of $\mathbb{R}$.

The following propositions are true:

(10) $x \in \prod G$ iff for every i holds $x(i) \in G(i)$.

(11) $\prod G$ is finite.

(12) For every non empty finite subset X of $\mathbb{R}$ there exists r_1 such that $r_1 \in X$ and for every x_1 such that $x_1 \in X$ holds $r_1 \geqslant x_1$.

(13) For every non empty finite subset X of $\mathbb{R}$ there exists l_1 such that $l_1 \in X$ and for every x_1 such that $x_1 \in X$ holds $l_1 \leqslant x_1$.

(14) There exist l_1, r_1 such that $l_1 \in G_1$ and $r_1 \in G_1$ and $l_1 < r_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ or $x_1 \not< r_1$.

(15) There exist l_1, r_1 such that $l_1 \in G_1$ and $r_1 \in G_1$ and $r_1 < l_1$ and for every x_1 such that $x_1 \in G_1$ holds $x_1 \not< r_1$ and $l_1 \not< x_1$.

Let us consider G_1. An element of $[\!:\mathbb{R}, \mathbb{R}:\!]$ is called a gap of G_1 if it satisfies the condition (Def. 6).

(Def. 6) There exist l_1, r_1 such that

(i) $\mathrm{it} = \langle l_1, r_1 \rangle$,

(ii) $l_1 \in G_1$,

(iii) $r_1 \in G_1$, and

(iv) $l_1 < r_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ or $x_1 \not< r_1$ or $r_1 < l_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ and $x_1 \not< r_1$.

The following propositions are true:

(16) $\langle l_1, r_1 \rangle$ is a gap of G_1 if and only if the following conditions are satisfied:

(i) $l_1 \in G_1$,

(ii) $r_1 \in G_1$, and

(iii) $l_1 < r_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ or $x_1 \not< r_1$ or $r_1 < l_1$ and for every x_1 such that $x_1 \in G_1$ holds $l_1 \not< x_1$ and $x_1 \not< r_1$.

(17) If $G_1 = \{l_1, r_1\}$, then $\langle l'_1, r'_1 \rangle$ is a gap of G_1 iff $l'_1 = l_1$ and $r'_1 = r_1$ or $l'_1 = r_1$ and $r'_1 = l_1$.

(18) If $x_1 \in G_1$, then there exists r_1 such that $\langle x_1, r_1 \rangle$ is a gap of G_1.

(19) If $x_1 \in G_1$, then there exists l_1 such that $\langle l_1, x_1 \rangle$ is a gap of G_1.

(20) If $\langle l_1, r_1 \rangle$ is a gap of G_1 and $\langle l_1, r'_1 \rangle$ is a gap of G_1, then $r_1 = r'_1$.

(21) If $\langle l_1, r_1 \rangle$ is a gap of G_1 and $\langle l'_1, r_1 \rangle$ is a gap of G_1, then $l_1 = l'_1$.

(22) If $r_1 < l_1$ and $\langle l_1, r_1 \rangle$ is a gap of G_1 and $r'_1 < l'_1$ and $\langle l'_1, r'_1 \rangle$ is a gap of G_1, then $l_1 = l'_1$ and $r_1 = r'_1$.

Let us consider d, l, r. The functor $\mathrm{cell}(l, r)$ yielding a non empty subset of $\mathcal{R}^d$ is defined as follows:

(Def. 7) $\mathrm{cell}(l, r) = \{x : \bigwedge_i (l(i) \leqslant x(i) \wedge x(i) \leqslant r(i)) \vee \bigvee_i (r(i) < l(i) \wedge (x(i) \leqslant r(i) \ \vee \ l(i) \leqslant x(i)))\}$.

We now state several propositions:

(23) $x \in \mathrm{cell}(l, r)$ iff for every i holds $l(i) \leqslant x(i)$ and $x(i) \leqslant r(i)$ or there exists i such that $r(i) < l(i)$ but $x(i) \leqslant r(i)$ or $l(i) \leqslant x(i)$.

(24) If for every i holds $l(i) \leqslant r(i)$, then $x \in \mathrm{cell}(l, r)$ iff for every i holds $l(i) \leqslant x(i)$ and $x(i) \leqslant r(i)$.

(25) If there exists i such that $r(i) < l(i)$, then $x \in \mathrm{cell}(l, r)$ iff there exists i such that $r(i) < l(i)$ but $x(i) \leqslant r(i)$ or $l(i) \leqslant x(i)$.

(26) $l \in \mathrm{cell}(l, r)$ and $r \in \mathrm{cell}(l, r)$.

(27) $\mathrm{cell}(x, x) = \{x\}$.

(28) If for every i holds $l'(i) \leqslant r'(i)$, then $\mathrm{cell}(l, r) \subseteq \mathrm{cell}(l', r')$ iff for every i holds $l'(i) \leqslant l(i)$ and $l(i) \leqslant r(i)$ and $r(i) \leqslant r'(i)$.

(29) If for every i holds $r(i) < l(i)$, then $\mathrm{cell}(l, r) \subseteq \mathrm{cell}(l', r')$ iff for every i holds $r(i) \leqslant r'(i)$ and $r'(i) < l'(i)$ and $l'(i) \leqslant l(i)$.

(30) Suppose for every i holds $l(i) \leqslant r(i)$ and for every i holds $r'(i) < l'(i)$. Then $\mathrm{cell}(l, r) \subseteq \mathrm{cell}(l', r')$ if and only if there exists i such that $r(i) \leqslant r'(i)$ or $l'(i) \leqslant l(i)$.

(31) If for every i holds $l(i) \leqslant r(i)$ or for every i holds $l(i) > r(i)$, then $\mathrm{cell}(l, r) = \mathrm{cell}(l', r')$ iff $l = l'$ and $r = r'$.

Let us consider d, G, k. Let us assume that $k \leqslant d$. The functor k-cells(G) yields a finite non empty subset of $2^{\mathcal{R}^d}$ and is defined by the condition (Def. 8).

(Def. 8) k-cells$(G) = \{\mathrm{cell}(l, r) : \bigvee_{X : \text{subset of } \mathrm{Seg}\, d} (\mathrm{card}\, X = k \ \wedge \ \bigwedge_i (i \in X \ \wedge \ l(i) < r(i) \ \wedge \ \langle l(i), r(i) \rangle$ is a gap of $G(i) \ \vee \ i \notin X \ \wedge \ l(i) = r(i) \ \wedge \ l(i) \in G(i))) \ \vee \ k = d \ \wedge \ \bigwedge_i (r(i) < l(i) \ \wedge \ \langle l(i), r(i) \rangle$ is a gap of $G(i))\}$.

We now state a number of propositions:

(32) Suppose $k \leqslant d$. Let A be a subset of $\mathcal{R}^d$. Then $A \in k$-cells(G) if and only if there exist l, r such that $A = \mathrm{cell}(l, r)$ but there exists a subset X of $\mathrm{Seg}\, d$ such that $\mathrm{card}\, X = k$ and for every i holds $i \in X$ and $l(i) < r(i)$ and $\langle l(i), r(i) \rangle$ is a gap of $G(i)$ or $i \notin X$ and $l(i) = r(i)$ and $l(i) \in G(i)$ or $k = d$ and for every i holds $r(i) < l(i)$ and $\langle l(i), r(i) \rangle$ is a gap of $G(i)$.

(33) Suppose $k \leqslant d$. Then $\mathrm{cell}(l, r) \in k$-cells$(G)$ if and only if one of the following conditions is satisfied:

(i) there exists a subset X of $\mathrm{Seg}\, d$ such that $\mathrm{card}\, X = k$ and for every i holds $i \in X$ and $l(i) < r(i)$ and $\langle l(i), r(i) \rangle$ is a gap of $G(i)$ or $i \notin X$ and $l(i) = r(i)$ and $l(i) \in G(i)$, or

(ii) $k = d$ and for every i holds $r(i) < l(i)$ and $\langle l(i), r(i) \rangle$ is a gap of $G(i)$.

(34) Suppose $k \leqslant d$ and $\mathrm{cell}(l, r) \in k$-cells$(G)$. Then

(i) for every i holds $l(i) < r(i)$ and $\langle l(i), r(i) \rangle$ is a gap of $G(i)$ or $l(i) = r(i)$ and $l(i) \in G(i)$, or

(ii) for every i holds $r(i) < l(i)$ and $\langle l(i), r(i) \rangle$ is a gap of $G(i)$.

(35) If $k \leqslant d$ and $\mathrm{cell}(l, r) \in k$-cells$(G)$, then for every i holds $l(i) \in G(i)$ and $r(i) \in G(i)$.

(36) If $k \leqslant d$ and $\mathrm{cell}(l, r) \in k$-cells$(G)$, then for every i holds $l(i) \leqslant r(i)$ or for every i holds $r(i) < l(i)$.

(37) For every subset A of $\mathcal{R}^d$ holds $A \in 0$-cells(G) iff there exists x such that $A = \mathrm{cell}(x, x)$ and for every i holds $x(i) \in G(i)$.

(38) $\mathrm{cell}(l, r) \in 0$-cells$(G)$ iff $l = r$ and for every i holds $l(i) \in G(i)$.

(39) Let A be a subset of $\mathcal{R}^d$. Then $A \in d$-cells(G) if and only if there exist l, r such that $A = \mathrm{cell}(l, r)$ but for every i holds $\langle l(i), r(i) \rangle$ is a gap of $G(i)$ but for every i holds $l(i) < r(i)$ or for every i holds $r(i) < l(i)$.

(40) $\mathrm{cell}(l, r) \in d$-cells$(G)$ iff for every i holds $\langle l(i), r(i) \rangle$ is a gap of $G(i)$ but for every i holds $l(i) < r(i)$ or for every i holds $r(i) < l(i)$.

(41) Suppose $d = d' + 1$. Let A be a subset of $\mathcal{R}^d$. Then $A \in d'$-cells(G) if and only if there exist l, r, i_0 such that $A = \mathrm{cell}(l, r)$ and $l(i_0) = r(i_0)$ and $l(i_0) \in G(i_0)$ and for every i such that $i \neq i_0$ holds $l(i) < r(i)$ and $\langle l(i), r(i) \rangle$ is a gap of $G(i)$.

(42) Suppose $d = d' + 1$. Then $\mathrm{cell}(l, r) \in d'$-cells$(G)$ if and only if there exists i_0 such that $l(i_0) = r(i_0)$ and $l(i_0) \in G(i_0)$ and for every i such that $i \neq i_0$ holds $l(i) < r(i)$ and $\langle l(i), r(i) \rangle$ is a gap of $G(i)$.

(43) Let A be a subset of $\mathcal{R}^d$. Then $A \in 1\text{-}\operatorname{cells}(G)$ if and only if there exist l, r, i_0 such that $A = \operatorname{cell}(l, r)$ and $l(i_0) < r(i_0)$ or $d = 1$ and $r(i_0) < l(i_0)$ and $\langle l(i_0), r(i_0)\rangle$ is a gap of $G(i_0)$ and for every i such that $i \neq i_0$ holds $l(i) = r(i)$ and $l(i) \in G(i)$.

(44) $\operatorname{cell}(l, r) \in 1\text{-}\operatorname{cells}(G)$ if and only if there exists i_0 such that $l(i_0) < r(i_0)$ or $d = 1$ and $r(i_0) < l(i_0)$ but $\langle l(i_0), r(i_0)\rangle$ is a gap of $G(i_0)$ but for every i such that $i \neq i_0$ holds $l(i) = r(i)$ and $l(i) \in G(i)$.

(45) Suppose $k \leqslant d$ and $k' \leqslant d$ and $\operatorname{cell}(l, r) \in k\text{-}\operatorname{cells}(G)$ and $\operatorname{cell}(l', r') \in k'\text{-}\operatorname{cells}(G)$ and $\operatorname{cell}(l, r) \subseteq \operatorname{cell}(l', r')$. Let given i. Then

(i) $l(i) = l'(i)$ and $r(i) = r'(i)$, or
(ii) $l(i) = l'(i)$ and $r(i) = l'(i)$, or
(iii) $l(i) = r'(i)$ and $r(i) = r'(i)$, or
(iv) $l(i) \leqslant r(i)$ and $r'(i) < l'(i)$ and $r'(i) \leqslant l(i)$ and $r(i) \leqslant l'(i)$.

(46) Suppose $k < k'$ and $k' \leqslant d$ and $\operatorname{cell}(l, r) \in k\text{-}\operatorname{cells}(G)$ and $\operatorname{cell}(l', r') \in k'\text{-}\operatorname{cells}(G)$ and $\operatorname{cell}(l, r) \subseteq \operatorname{cell}(l', r')$. Then there exists i such that $l(i) = l'(i)$ and $r(i) = l'(i)$ or $l(i) = r'(i)$ and $r(i) = r'(i)$.

(47) Let X, X' be subsets of $\operatorname{Seg} d$. Suppose that

(i) $\operatorname{cell}(l, r) \subseteq \operatorname{cell}(l', r')$,
(ii) for every i holds $i \in X$ and $l(i) < r(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$ or $i \notin X$ and $l(i) = r(i)$ and $l(i) \in G(i)$, and
(iii) for every i holds $i \in X'$ and $l'(i) < r'(i)$ and $\langle l'(i), r'(i)\rangle$ is a gap of $G(i)$ or $i \notin X'$ and $l'(i) = r'(i)$ and $l'(i) \in G(i)$.

Then

(iv) $X \subseteq X'$,
(v) for every i such that $i \in X$ or $i \notin X'$ holds $l(i) = l'(i)$ and $r(i) = r'(i)$, and
(vi) for every i such that $i \notin X$ and $i \in X'$ holds $l(i) = l'(i)$ and $r(i) = l'(i)$ or $l(i) = r'(i)$ and $r(i) = r'(i)$.

Let us consider d, G, k. A k-cell of G is an element of $k\text{-}\operatorname{cells}(G)$.

Let us consider d, G, k. A k-chain of G is a subset of $k\text{-}\operatorname{cells}(G)$.

Let us consider d, G, k. The functor $0_k G$ yields a k-chain of G and is defined as follows:

(Def. 9) $0_k G = \emptyset$.

Let us consider d, G. The functor ΩG yielding a d-chain of G is defined as follows:

(Def. 10) $\Omega G = d\text{-}\operatorname{cells}(G)$.

Let us consider d, G, k and let C_1, C_2 be k-chains of G. Then $C_1 \dot{-} C_2$ is a k-chain of G. We introduce $C_1 + C_2$ as a synonym of $C_1 \dot{-} C_2$.

Let us consider d, G. The infinite cell of G yielding a d-cell of G is defined by:

(Def. 11) There exist l, r such that the infinite cell of $G = \mathrm{cell}(l, r)$ and for every i holds $r(i) < l(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$.

We now state two propositions:

(48) If $\mathrm{cell}(l, r)$ is a d-cell of G, then $\mathrm{cell}(l, r) =$ the infinite cell of G iff for every i holds $r(i) < l(i)$.

(49) $\mathrm{cell}(l, r) =$ the infinite cell of G iff for every i holds $r(i) < l(i)$ and $\langle l(i), r(i)\rangle$ is a gap of $G(i)$.

The scheme *ChainInd* deals with a non zero natural number $\mathcal{A}$, a $\mathcal{A}$-dimensional grating $\mathcal{B}$, a natural number $\mathcal{C}$, a $\mathcal{C}$-chain $\mathcal{D}$ of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{D}]$

provided the parameters have the following properties:

- $\mathcal{P}[0_{\mathcal{C}}\mathcal{B}]$,
- For every $\mathcal{C}$-cell A of $\mathcal{B}$ such that $A \in \mathcal{D}$ holds $\mathcal{P}[\{A\}]$, and
- For all $\mathcal{C}$-chains C_1, C_2 of $\mathcal{B}$ such that $C_1 \subseteq \mathcal{D}$ and $C_2 \subseteq \mathcal{D}$ and $\mathcal{P}[C_1]$ and $\mathcal{P}[C_2]$ holds $\mathcal{P}[C_1 + C_2]$.

Let us consider d, G, k and let A be a k-cell of G. The functor $A^\star$ yields a $k+1$-chain of G and is defined by:

(Def. 12) $A^\star = \{B; B$ ranges over $k+1$-cells of G: $A \subseteq B\}$.

Next we state the proposition

(50) For every k-cell A of G and for every $k+1$-cell B of G holds $B \in A^\star$ iff $A \subseteq B$.

Let us consider d, G, k and let C be a $k+1$-chain of G. The functor ∂C yielding a k-chain of G is defined as follows:

(Def. 13) $\partial C = \{A; A$ ranges over k-cells of G: $k+1 \leqslant d \ \wedge\ \mathrm{card}(A^\star \cap C)$is odd$\}$.

We introduce $\dot{C}$ as a synonym of ∂C.

Let us consider d, G, k, let C be a $k+1$-chain of G, and let C' be a k-chain of G. We say that C' bounds C if and only if:

(Def. 14) $C' = \partial C$.

The following propositions are true:

(51) For every k-cell A of G and for every $k+1$-chain C of G holds $A \in \partial C$ iff $k+1 \leqslant d$ and $\mathrm{card}(A^\star \cap C)$ is odd.

(52) If $k+1 > d$, then for every $k+1$-chain C of G holds $\partial C = 0_k G$.

(53) If $k+1 \leqslant d$, then for every k-cell A of G and for every $k+1$-cell B of G holds $A \in \partial\{B\}$ iff $A \subseteq B$.

(54) If $d = d'+1$, then for every d'-cell A of G holds $\mathrm{card}\, A^\star = 2$.

(55) For every d-dimensional grating G and for every $0+1$-cell B of G holds $\mathrm{card}\, \partial\{B\} = 2$.

(56) $\Omega G = (0_d G)^{\mathrm{c}}$ and $0_d G = (\Omega G)^{\mathrm{c}}$.

(57) For every k-chain C of G holds $C + 0_k G = C$.

(58) For every k-chain C of G holds $C + C = 0_k G$.

(59) For every d-chain C of G holds $C^{\mathrm{c}} = C + \Omega G$.

(60) $\partial 0_{k+1} G = 0_k G$.

(61) For every $d' + 1$-dimensional grating G holds $\partial \Omega G = 0_{d'} G$.

(62) For all $k + 1$-chains C_1, C_2 of G holds $\partial(C_1 + C_2) = \partial C_1 + \partial C_2$.

(63) For every $d' + 1$-dimensional grating G and for every $d' + 1$-chain C of G holds $\partial(C^{\mathrm{c}}) = \partial C$.

(64) For every $k + 1 + 1$-chain C of G holds $\partial\partial C = 0_k G$.

Let us consider d, G, k. A k-chain of G is called a k-cycle of G if:

(Def. 15) $k = 0$ and card it is even or there exists k' such that $k = k' + 1$ and there exists a $k' + 1$-chain C of G such that $C =$ it and $\partial C = 0_{k'} G$.

One can prove the following propositions:

(65) For every $k + 1$-chain C of G holds C is a $k + 1$-cycle of G iff $\partial C = 0_k G$.

(66) If $k > d$, then every k-chain of G is a k-cycle of G.

(67) For every 0-chain C of G holds C is a 0-cycle of G iff $\operatorname{card} C$ is even.

Let us consider d, G, k and let C be a $k + 1$-cycle of G. Then ∂C can be characterized by the condition:

(Def. 16) $\partial C = 0_k G$.

Let us consider d, G, k. Then $0_k G$ is a k-cycle of G.

Let us consider d, G. Then ΩG is a d-cycle of G.

Let us consider d, G, k and let C_1, C_2 be k-cycles of G. Then $C_1 \dot{-} C_2$ is a k-cycle of G. We introduce $C_1 + C_2$ as a synonym of $C_1 \dot{-} C_2$.

We now state the proposition

(68) For every d-cycle C of G holds C^{c} is a d-cycle of G.

Let us consider d, G, k and let C be a $k+1$-chain of G. Then ∂C is a k-cycle of G.

3. Groups and Homomorphisms

Let us consider d, G, k. The functor k-$\mathrm{Chains}(G)$ yields a strict Abelian group and is defined by the conditions (Def. 17).

(Def. 17)(i) The carrier of k-$\mathrm{Chains}(G) = 2^{k\text{-}\mathrm{cells}(G)}$,

(ii) $0_{k\text{-}\mathrm{Chains}(G)} = 0_k G$, and

(iii) for all elements A, B of k-$\mathrm{Chains}(G)$ and for all k-chains A', B' of G such that $A = A'$ and $B = B'$ holds $A + B = A' + B'$.

Let us consider d, G, k. A k-grchain of G is an element of k-$\mathrm{Chains}(G)$.

One can prove the following proposition

(69) For every set x holds x is a k-chain of G iff x is a k-grchain of G.

Let us consider d, G, k. The functor ∂ yielding a homomorphism from $(k+1)$-Chains(G) to k-Chains(G) is defined by:

(Def. 18) For every element A of $(k+1)$-Chains(G) and for every $k+1$-chain A' of G such that $A = A'$ holds $\partial(A) = \partial A'$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[6] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[12] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[13] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[14] Michał Muzalewski. Opposite rings, modules and their morphisms. *Formalized Mathematics*, 3(**1**):57–65, 1992.
[15] M. H. A. Newman. *Elements of the Topology of Plane Sets of Points.* Cambridge University Press, 1951.
[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[17] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Formalized Mathematics*, 6(**3**):335–338, 1997.
[18] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics.*
[19] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received January 27, 2003