FORMALIZED MATHEMATICS
Volume 11, Number 2, 2003
University of Białystok

The Underlying Principle of Dijkstra's Shortest Path Algorithm

Jing-Chao Chen
Donghua University
Shanghai

Yatsuka Nakamura
Shinshu University
Nagano

Summary. A path from a source vertex v to a target vertex u is said to be a shortest path if its total cost is minimum among all v-to-u paths. Dijkstra's algorithm is a classic shortest path algorithm, which is described in many textbooks. To justify its correctness (whose rigorous proof will be given in the next article), it is necessary to clarify its underlying principle. For this purpose, the article justifies the following basic facts, which are the core of Dijkstra's algorithm.

- A graph is given, its vertex set is denoted by V. Assume U is the subset of V, and if a path p from s to t is the shortest among the set of paths, each of which passes through only the vertices in U, except the source and sink, and its source and sink is s and in V, respectively, then p is a shortest path from s to t in the graph, and for any subgraph which contains at least U, it is also the shortest.
- Let $p(s,x,U)$ denote the shortest path from s to x in a subgraph whose the vertex set is the union of $\{s,x\}$ and U, and cost (p) denote the cost of path $p(s,x,U)$, $\mathrm{cost}(x,y)$ the cost of the edge from x to y. Give $p(s,x,U)$, $q(s,y,U)$ and $r(s,y,U\cup\{x\})$. If $\mathrm{cost}(p)=\min\{\mathrm{cost}(w) : w(s,t,U)\wedge t\in V\}$, then we have

$$\mathrm{cost}(r)=\min(\mathrm{cost}(p)+\mathrm{cost}(x,y),\mathrm{cost}(q)).$$

This is the well-known triangle comparison of Dijkstra's algorithm.

MML Identifier: GRAPH_5.

The articles [14], [16], [13], [17], [5], [3], [4], [15], [1], [8], [9], [2], [10], [6], [12], [7], and [11] provide the terminology and notation for this paper.

ISSN 1426–2630

1. Preliminaries

We follow the rules: n, m, i, j, k denote natural numbers, x, y, e, X, V, U denote sets, and W, f, g denote functions.

Let f be a finite function. Observe that $\mathrm{rng}\, f$ is finite.

One can prove the following two propositions:

(1) For every finite function f holds $\mathrm{card}\, \mathrm{rng}\, f \leqslant \mathrm{card}\, \mathrm{dom}\, f$.

(2) If $\mathrm{rng}\, f \subseteq \mathrm{rng}\, g$ and $x \in \mathrm{dom}\, f$, then there exists y such that $y \in \mathrm{dom}\, g$ and $f(x) = g(y)$.

The scheme *LambdaAB* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a function f such that $\mathrm{dom}\, f = \mathcal{A}$ and for every element x of $\mathcal{B}$ such that $x \in \mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

The following propositions are true:

(3) Let D be a finite set, n be a natural number, and X be a set. If $X = \{x; x$ ranges over elements of D^*: $1 \leqslant \mathrm{len}\, x \ \wedge\ \mathrm{len}\, x \leqslant n\}$, then X is finite.

(4) Let D be a finite set, n be a natural number, and X be a set. If $X = \{x; x$ ranges over elements of D^*: $\mathrm{len}\, x \leqslant n\}$, then X is finite.

(5) For every finite set D holds $\mathrm{card}\, D \neq 0$ iff $D \neq \emptyset$.

(6) Let D be a finite set and k be a natural number. Suppose $\mathrm{card}\, D = k+1$. Then there exists an element x of D and there exists a subset C of D such that $D = C \cup \{x\}$ and $\mathrm{card}\, C = k$.

(7) For every finite set D such that $\mathrm{card}\, D = 1$ there exists an element x of D such that $D = \{x\}$.

The scheme *MinValue* deals with a non empty finite set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a real number, and states that:

> There exists an element x of $\mathcal{A}$ such that for every element y of $\mathcal{A}$ holds $\mathcal{F}(x) \leqslant \mathcal{F}(y)$

for all values of the parameters.

Let D be a set and let X be a non empty subset of D^*. We see that the element of X is a finite sequence of elements of D.

2. Additional Properties of Finite Sequences

In the sequel p, q are finite sequences.

One can prove the following propositions:

(8) $p \neq \emptyset$ iff $\mathrm{len}\, p \geqslant 1$.

(9) For all n, m such that $1 \leqslant n$ and $n < m$ and $m \leqslant \mathrm{len}\, p$ holds $p(n) \neq p(m)$ iff p is one-to-one.

(10) For all n, m such that $1 \leqslant n$ and $n < m$ and $m \leqslant \operatorname{len} p$ holds $p(n) \neq p(m)$ iff $\operatorname{card} \operatorname{rng} p = \operatorname{len} p$.

In the sequel G denotes a graph and p_1, q_1 denote finite sequences of elements of the edges of G.

Next we state two propositions:

(11) If $i \in \operatorname{dom} p_1$, then (the source of G)($p_1(i)$) $\in$ the vertices of G and (the target of G)($p_1(i)$) $\in$ the vertices of G.

(12) If $q \frown \langle x \rangle$ is one-to-one and $\operatorname{rng}(q \frown \langle x \rangle) \subseteq \operatorname{rng} p$, then there exist finite sequences p_2, p_3 such that $p = p_2 \frown \langle x \rangle \frown p_3$ and $\operatorname{rng} q \subseteq \operatorname{rng}(p_2 \frown p_3)$.

3. Additional Properties of Chains and Oriented Paths

One can prove the following three propositions:

(13) If $p \frown q$ is a chain of G, then p is a chain of G and q is a chain of G.

(14) If $p \frown q$ is an oriented chain of G, then p is an oriented chain of G and q is an oriented chain of G.

(15) Let p, q be oriented chains of G. Suppose (the target of G)($p(\operatorname{len} p)$) = (the source of G)($q(1)$). Then $p \frown q$ is an oriented chain of G.

4. Additional Properties of Acyclic Oriented Paths

The following propositions are true:

(16) $\emptyset$ is a Simple oriented chain of G.

(17) Suppose $p \frown q$ is a Simple oriented chain of G. Then p is a Simple oriented chain of G and q is a Simple oriented chain of G.

(18) If $\operatorname{len} p_1 = 1$, then p_1 is a Simple oriented chain of G.

(19) Let p be a Simple oriented chain of G and q be a finite sequence of elements of the edges of G. Suppose that
  (i) $\operatorname{len} p \geqslant 1$,
  (ii) $\operatorname{len} q = 1$,
  (iii) (the source of G)($q(1)$) = (the target of G)($p(\operatorname{len} p)$),
  (iv) (the source of G)($p(1)$) $\neq$ (the target of G)($p(\operatorname{len} p)$), and
  (v) it is not true that there exists k such that $1 \leqslant k$ and $k \leqslant \operatorname{len} p$ and (the target of G)($p(k)$) = (the target of G)($q(1)$).

  Then $p \frown q$ is a Simple oriented chain of G.

(20) Every Simple oriented chain of G is one-to-one.

5. The Set of the Vertices On a Path or an Edge

Let G be a graph and let e be an element of the edges of G. The functor $\mathrm{vertices}\, e$ is defined as follows:

(Def. 1) $\mathrm{vertices}\, e = \{(\text{the source of } G)(e), (\text{the target of } G)(e)\}$.

Let us consider G, p_1. The functor $\mathrm{vertices}\, p_1$ yields a subset of the vertices of G and is defined by:

(Def. 2) $\mathrm{vertices}\, p_1 = \{v; v$ ranges over vertices of G: $\bigvee_i\ (i \in \mathrm{dom}\, p_1 \ \wedge\ v \in \mathrm{vertices}((p_1)_i))\}$.

We now state several propositions:

(21) Let p be a Simple oriented chain of G. Suppose $p = p_1 \frown q_1$ and $\mathrm{len}\, p_1 \geqslant 1$ and $\mathrm{len}\, q_1 \geqslant 1$ and $(\text{the source of } G)(p(1)) \neq (\text{the target of } G)(p(\mathrm{len}\, p))$. Then $(\text{the source of } G)(p(1)) \notin \mathrm{vertices}\, q_1$ and $(\text{the target of } G)(p(\mathrm{len}\, p)) \notin \mathrm{vertices}\, p_1$.

(22) $\mathrm{vertices}\, p_1 \subseteq V$ iff for every i such that $i \in \mathrm{dom}\, p_1$ holds $\mathrm{vertices}((p_1)_i) \subseteq V$.

(23) Suppose $\mathrm{vertices}\, p_1 \not\subseteq V$. Then there exists a natural number i and there exist finite sequences q, r of elements of the edges of G such that $i + 1 \leqslant \mathrm{len}\, p_1$ and $\mathrm{vertices}((p_1)_{i+1}) \not\subseteq V$ and $\mathrm{len}\, q = i$ and $p_1 = q \frown r$ and $\mathrm{vertices}\, q \subseteq V$.

(24) If $\mathrm{rng}\, q_1 \subseteq \mathrm{rng}\, p_1$, then $\mathrm{vertices}\, q_1 \subseteq \mathrm{vertices}\, p_1$.

(25) If $\mathrm{rng}\, q_1 \subseteq \mathrm{rng}\, p_1$ and $\mathrm{vertices}\, p_1 \setminus X \subseteq V$, then $\mathrm{vertices}\, q_1 \setminus X \subseteq V$.

(26) If $\mathrm{vertices}(p_1 \frown q_1) \setminus X \subseteq V$, then $\mathrm{vertices}\, p_1 \setminus X \subseteq V$ and $\mathrm{vertices}\, q_1 \setminus X \subseteq V$.

In the sequel v, v_1, v_2, v_3 denote elements of the vertices of G.

One can prove the following four propositions:

(27) For every element e of the edges of G such that $v = (\text{the source of } G)(e)$ or $v = (\text{the target of } G)(e)$ holds $v \in \mathrm{vertices}\, e$.

(28) If $i \in \mathrm{dom}\, p_1$ and if $v = (\text{the source of } G)(p_1(i))$ or $v = (\text{the target of } G)(p_1(i))$, then $v \in \mathrm{vertices}\, p_1$.

(29) If $\mathrm{len}\, p_1 = 1$, then $\mathrm{vertices}\, p_1 = \mathrm{vertices}((p_1)_1)$.

(30) $\mathrm{vertices}\, p_1 \subseteq \mathrm{vertices}(p_1 \frown q_1)$ and $\mathrm{vertices}\, q_1 \subseteq \mathrm{vertices}(p_1 \frown q_1)$.

In the sequel p, q are oriented chains of G.

Next we state two propositions:

(31) If $p = q \frown p_1$ and $\mathrm{len}\, q \geqslant 1$ and $\mathrm{len}\, p_1 = 1$, then $\mathrm{vertices}\, p = \mathrm{vertices}\, q \cup \{(\text{the target of } G)(p_1(1))\}$.

(32) If $v \neq (\text{the source of } G)(p(1))$ and $v \in \mathrm{vertices}\, p$, then there exists i such that $1 \leqslant i$ and $i \leqslant \mathrm{len}\, p$ and $v = (\text{the target of } G)(p(i))$.

6. Directed Paths between Two Vertices

Let us consider G, p, v_1, v_2. We say that p is oriented path from v_1 to v_2 if and only if:

(Def. 3) $p \neq \emptyset$ and (the source of G)$(p(1)) = v_1$ and (the target of G)$(p(\operatorname{len} p)) = v_2$.

Let us consider G, v_1, v_2, p, V. We say that p is oriented path from v_1 to v_2 in V if and only if:

(Def. 4) p is oriented path from v_1 to v_2 and $\operatorname{vertices} p \setminus \{v_2\} \subseteq V$.

Let G be a graph and let v_1, v_2 be elements of the vertices of G. The functor $\mathrm{OrientedPaths}(v_1, v_2)$ yields a subset of (the edges of G)* and is defined by:

(Def. 5) $\mathrm{OrientedPaths}(v_1, v_2) = \{p; p$ ranges over oriented chains of G: p is oriented path from v_1 to $v_2\}$.

Next we state several propositions:

(33) If p is oriented path from v_1 to v_2, then $v_1 \in \operatorname{vertices} p$ and $v_2 \in \operatorname{vertices} p$.

(34) $x \in \mathrm{OrientedPaths}(v_1, v_2)$ iff there exists p such that $p = x$ and p is oriented path from v_1 to v_2.

(35) If p is oriented path from v_1 to v_2 in V and $v_1 \neq v_2$, then $v_1 \in V$.

(36) If p is oriented path from v_1 to v_2 in V and $V \subseteq U$, then p is oriented path from v_1 to v_2 in U.

(37) Suppose $\operatorname{len} p \geqslant 1$ and p is oriented path from v_1 to v_2 and $p_1(1)$ orientedly joins v_2, v_3 and $\operatorname{len} p_1 = 1$. Then there exists q such that $q = p \frown p_1$ and q is oriented path from v_1 to v_3.

(38) Suppose $q = p \frown p_1$ and $\operatorname{len} p \geqslant 1$ and $\operatorname{len} p_1 = 1$ and p is oriented path from v_1 to v_2 in V and $p_1(1)$ orientedly joins v_2, v_3. Then q is oriented path from v_1 to v_3 in $V \cup \{v_2\}$.

7. Acyclic (or Simple) Paths

Let G be a graph, let p be an oriented chain of G, and let v_1, v_2 be elements of the vertices of G. We say that p is acyclic path from v_1 to v_2 if and only if:

(Def. 6) p is Simple and oriented path from v_1 to v_2.

Let G be a graph, let p be an oriented chain of G, let v_1, v_2 be elements of the vertices of G, and let V be a set. We say that p is acyclic path from v_1 to v_2 in V if and only if:

(Def. 7) p is Simple and oriented path from v_1 to v_2 in V.

Let G be a graph and let v_1, v_2 be elements of the vertices of G. The functor $\mathrm{AcyclicPaths}(v_1, v_2)$ yielding a subset of (the edges of G)* is defined as follows:

(Def. 8) AcyclicPaths$(v_1, v_2) = \{p; p$ ranges over Simple oriented chains of G: p is acyclic path from v_1 to $v_2\}$.

Let G be a graph, let v_1, v_2 be elements of the vertices of G, and let V be a set. The functor AcyclicPaths(v_1, v_2, V) yielding a subset of (the edges of $G)^*$ is defined as follows:

(Def. 9) AcyclicPaths$(v_1, v_2, V) = \{p; p$ ranges over Simple oriented chains of G: p is acyclic path from v_1 to v_2 in $V\}$.

Let G be a graph and let p be an oriented chain of G. The functor AcyclicPaths(p) yielding a subset of (the edges of $G)^*$ is defined by the condition (Def. 10).

(Def. 10) AcyclicPaths$(p) = \{q; q$ ranges over Simple oriented chains of G: $q \neq \emptyset \wedge$ (the source of $G)(q(1)) =$ (the source of $G)(p(1)) \wedge$ (the target of $G)(q(\operatorname{len} q)) =$ (the target of $G)(p(\operatorname{len} p)) \wedge \operatorname{rng} q \subseteq \operatorname{rng} p\}$.

Let G be a graph. The functor AcyclicPaths(G) yields a subset of (the edges of $G)^*$ and is defined as follows:

(Def. 11) AcyclicPaths$(G) = \{q : q$ ranges over Simple oriented chains of $G\}$.

The following propositions are true:

(39) If $p = \emptyset$, then p is not acyclic path from v_1 to v_2.

(40) If p is acyclic path from v_1 to v_2, then p is oriented path from v_1 to v_2.

(41) AcyclicPaths$(v_1, v_2) \subseteq$ OrientedPaths(v_1, v_2).

(42) AcyclicPaths$(p) \subseteq$ AcyclicPaths(G).

(43) AcyclicPaths$(v_1, v_2) \subseteq$ AcyclicPaths(G).

(44) If p is oriented path from v_1 to v_2, then AcyclicPaths$(p) \subseteq$ AcyclicPaths(v_1, v_2).

(45) If p is oriented path from v_1 to v_2 in V, then AcyclicPaths$(p) \subseteq$ AcyclicPaths(v_1, v_2, V).

(46) If $q \in$ AcyclicPaths(p), then $\operatorname{len} q \leqslant \operatorname{len} p$.

(47) If p is oriented path from v_1 to v_2, then AcyclicPaths$(p) \neq \emptyset$ and AcyclicPaths$(v_1, v_2) \neq \emptyset$.

(48) If p is oriented path from v_1 to v_2 in V, then AcyclicPaths$(p) \neq \emptyset$ and AcyclicPaths$(v_1, v_2, V) \neq \emptyset$.

(49) AcyclicPaths$(v_1, v_2, V) \subseteq$ AcyclicPaths(G).

8. Weight Graphs and Their Basic Properties

The subset $\mathbb{R}_{\geqslant 0}$ of $\mathbb{R}$ is defined by:

(Def. 12) $\mathbb{R}_{\geqslant 0} = \{r; r$ ranges over real numbers: $r \geqslant 0\}$.

Let us mention that $\mathbb{R}_{\geqslant 0}$ is non empty.

Let G be a graph and let W be a function. We say that W is nonnegative weight of G if and only if:

(Def. 13) W is a function from the edges of G into $\mathbb{R}_{\geqslant 0}$.

Let G be a graph and let W be a function. We say that W is weight of G if and only if:

(Def. 14) W is a function from the edges of G into $\mathbb{R}$.

Let G be a graph, let p be a finite sequence of elements of the edges of G, and let W be a function. Let us assume that W is weight of G. The functor $\mathrm{RealSequence}(p, W)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 15) $\mathrm{dom}\, p = \mathrm{dom}\,\mathrm{RealSequence}(p, W)$ and for every natural number i such that $i \in \mathrm{dom}\, p$ holds $(\mathrm{RealSequence}(p, W))(i) = W(p(i))$.

Let G be a graph, let p be a finite sequence of elements of the edges of G, and let W be a function. The functor $\mathrm{cost}(p, W)$ yields a real number and is defined as follows:

(Def. 16) $\mathrm{cost}(p, W) = \sum \mathrm{RealSequence}(p, W)$.

Next we state a number of propositions:

(50) If W is nonnegative weight of G, then W is weight of G.

(51) Let f be a finite sequence of elements of $\mathbb{R}$. Suppose W is nonnegative weight of G and $f = \mathrm{RealSequence}(p_1, W)$. Let given i. If $i \in \mathrm{dom}\, f$, then $f(i) \geqslant 0$.

(52) If $\mathrm{rng}\, q_1 \subseteq \mathrm{rng}\, p_1$ and W is weight of G and $i \in \mathrm{dom}\, q_1$, then there exists j such that $j \in \mathrm{dom}\, p_1$ and $(\mathrm{RealSequence}(p_1, W))(j) = (\mathrm{RealSequence}(q_1, W))(i)$.

(53) If $\mathrm{len}\, q_1 = 1$ and $\mathrm{rng}\, q_1 \subseteq \mathrm{rng}\, p_1$ and W is nonnegative weight of G, then $\mathrm{cost}(q_1, W) \leqslant \mathrm{cost}(p_1, W)$.

(54) If W is nonnegative weight of G, then $\mathrm{cost}(p_1, W) \geqslant 0$.

(55) If $p_1 = \emptyset$ and W is weight of G, then $\mathrm{cost}(p_1, W) = 0$.

(56) Let D be a non empty finite subset of $(\text{the edges of } G)^*$. If $D = \mathrm{AcyclicPaths}(v_1, v_2)$, then there exists p_1 such that $p_1 \in D$ and for every q_1 such that $q_1 \in D$ holds $\mathrm{cost}(p_1, W) \leqslant \mathrm{cost}(q_1, W)$.

(57) Let D be a non empty finite subset of $(\text{the edges of } G)^*$. If $D = \mathrm{AcyclicPaths}(v_1, v_2, V)$, then there exists p_1 such that $p_1 \in D$ and for every q_1 such that $q_1 \in D$ holds $\mathrm{cost}(p_1, W) \leqslant \mathrm{cost}(q_1, W)$.

(58) If W is weight of G, then $\mathrm{cost}(p_1 \frown q_1, W) = \mathrm{cost}(p_1, W) + \mathrm{cost}(q_1, W)$.

(59) If q_1 is one-to-one and $\mathrm{rng}\, q_1 \subseteq \mathrm{rng}\, p_1$ and W is nonnegative weight of G, then $\mathrm{cost}(q_1, W) \leqslant \mathrm{cost}(p_1, W)$.

(60) If $p_1 \in \mathrm{AcyclicPaths}(p)$ and W is nonnegative weight of G, then $\mathrm{cost}(p_1, W) \leqslant \mathrm{cost}(p, W)$.

9. Shortest Paths and Their Basic Properties

Let G be a graph, let v_1, v_2 be vertices of G, let p be an oriented chain of G, and let W be a function. We say that p is shortest path from v_1 to v_2 in W if and only if the conditions (Def. 17) are satisfied.

(Def. 17)(i) p is oriented path from v_1 to v_2, and

(ii) for every oriented chain q of G such that q is oriented path from v_1 to v_2 holds $\mathrm{cost}(p, W) \leqslant \mathrm{cost}(q, W)$.

Let G be a graph, let v_1, v_2 be vertices of G, let p be an oriented chain of G, let V be a set, and let W be a function. We say that p is shortest path from v_1 to v_2 in V w.r.t. W if and only if the conditions (Def. 18) are satisfied.

(Def. 18)(i) p is oriented path from v_1 to v_2 in V, and

(ii) for every oriented chain q of G such that q is oriented path from v_1 to v_2 in V holds $\mathrm{cost}(p, W) \leqslant \mathrm{cost}(q, W)$.

10. Basic Properties of a Graph with Finite Vertices

For simplicity, we adopt the following rules: G is a finite graph, p_4 is a Simple oriented chain of G, P, Q are oriented chains of G, v_1, v_2, v_3 are elements of the vertices of G, and p_1, q_1 are finite sequences of elements of the edges of G.

One can prove the following two propositions:

(61) $\mathrm{len}\, p_4 \leqslant$ the number of vertices of G.

(62) $\mathrm{len}\, p_4 \leqslant$ the number of edges of G.

Let us consider G. Note that $\mathrm{AcyclicPaths}(G)$ is finite.

Let us consider G, P. Note that $\mathrm{AcyclicPaths}(P)$ is finite.

Let us consider G, v_1, v_2. One can verify that $\mathrm{AcyclicPaths}(v_1, v_2)$ is finite.

Let us consider G, v_1, v_2, V. Observe that $\mathrm{AcyclicPaths}(v_1, v_2, V)$ is finite.

We now state four propositions:

(63) If $\mathrm{AcyclicPaths}(v_1, v_2) \neq \emptyset$, then there exists p_1 such that $p_1 \in \mathrm{AcyclicPaths}(v_1, v_2)$ and for every q_1 such that $q_1 \in \mathrm{AcyclicPaths}(v_1, v_2)$ holds $\mathrm{cost}(p_1, W) \leqslant \mathrm{cost}(q_1, W)$.

(64) If $\mathrm{AcyclicPaths}(v_1, v_2, V) \neq \emptyset$, then there exists p_1 such that $p_1 \in \mathrm{AcyclicPaths}(v_1, v_2, V)$ and for every q_1 such that $q_1 \in \mathrm{AcyclicPaths}(v_1, v_2, V)$ holds $\mathrm{cost}(p_1, W) \leqslant \mathrm{cost}(q_1, W)$.

(65) If P is oriented path from v_1 to v_2 and W is nonnegative weight of G, then there exists a Simple oriented chain of G which is shortest path from v_1 to v_2 in W.

(66) Suppose P is oriented path from v_1 to v_2 in V and W is nonnegative weight of G. Then there exists a Simple oriented chain of G which is shortest path from v_1 to v_2 in V w.r.t. W.

11. Three Basic Theorems for Dijkstra's Shortest Path Algorithm

We now state two propositions:

(67) Suppose that

(i) W is nonnegative weight of G,

(ii) P is shortest path from v_1 to v_2 in V w.r.t. W,

(iii) $v_1 \neq v_2$, and

(iv) for all Q, v_3 such that $v_3 \notin V$ and Q is shortest path from v_1 to v_3 in V w.r.t. W holds $\text{cost}(P, W) \leqslant \text{cost}(Q, W)$.

Then P is shortest path from v_1 to v_2 in W.

(68) Suppose that

(i) W is nonnegative weight of G,

(ii) P is shortest path from v_1 to v_2 in V w.r.t. W,

(iii) $v_1 \neq v_2$,

(iv) $V \subseteq U$, and

(v) for all Q, v_3 such that $v_3 \notin V$ and Q is shortest path from v_1 to v_3 in V w.r.t. W holds $\text{cost}(P, W) \leqslant \text{cost}(Q, W)$.

Then P is shortest path from v_1 to v_2 in U w.r.t. W.

Let G be a graph, let p_1 be a finite sequence of elements of the edges of G, let V be a set, let v_1 be a vertex of G, and let W be a function. We say that p_1 is longest in shortest path from v_1 in V w.r.t. W if and only if the condition (Def. 19) is satisfied.

(Def. 19) Let v be a vertex of G. Suppose $v \in V$ and $v \neq v_1$. Then there exists an oriented chain q of G such that q is shortest path from v_1 to v in V w.r.t. W and $\text{cost}(q, W) \leqslant \text{cost}(p_1, W)$.

One can prove the following proposition

(69) Let G be a finite oriented graph, P, Q, R be oriented chains of G, and v_1, v_2, v_3 be elements of the vertices of G such that $e \in$ the edges of G and W is nonnegative weight of G and $\text{len}\, P \geqslant 1$ and P is shortest path from v_1 to v_2 in V w.r.t. W and $v_1 \neq v_2$ and $v_1 \neq v_3$ and $R = P ^\frown \langle e \rangle$ and Q is shortest path from v_1 to v_3 in V w.r.t. W and e orientedly joins v_2, v_3 and P is longest in shortest path from v_1 in V w.r.t. W. Then

(i) if $\text{cost}(Q, W) \leqslant \text{cost}(R, W)$, then Q is shortest path from v_1 to v_3 in $V \cup \{v_2\}$ w.r.t. W, and

(ii) if $\text{cost}(Q, W) \geqslant \text{cost}(R, W)$, then R is shortest path from v_1 to v_3 in $V \cup \{v_2\}$ w.r.t. W.

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Received January 7, 2003