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Solving Roots of Polynomial Equation of Degree 4 with Real Coefficients

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Summary. In this paper, we describe the definition of the fourth degree algebraic equations and their properties. We clarify the relation between the four roots of this equation and its coefficient. Also, the form of these roots for various conditions is discussed. This solution is known as the Cardano solution.

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The articles [3], [4], [1], and [2] provide the notation and terminology for this paper.

Let a, b, c, d, e, x be real numbers. The functor $\mathrm{Four}(a,b,c,d,e,x)$ is defined by:

(Def. 1) $\mathrm{Four}(a,b,c,d,e,x) = a \cdot x^4 + b \cdot x^3 + c \cdot x^{\mathbf{2}} + d \cdot x + e$.

Let a, b, c, d, e, x be real numbers. Note that $\mathrm{Four}(a,b,c,d,e,x)$ is real.

One can prove the following propositions:

(1) Let a, c, e, x be real numbers. Suppose $a \neq 0$ and $e \neq 0$ and $c^{\mathbf{2}} - 4 \cdot a \cdot e > 0$. Suppose $\mathrm{Four}(a,0,c,0,e,x) = 0$. Then $x \neq 0$ but $x = \sqrt{\frac{-c+\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$ or $x = \sqrt{\frac{-c-\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$ or $x = -\sqrt{\frac{-c+\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$ or $x = -\sqrt{\frac{-c-\sqrt{\Delta(a,c,e)}}{2 \cdot a}}$.

(2) Let a, b, c, x, y be real numbers. Suppose $a \neq 0$ and $y = x + \frac{1}{x}$. If $\mathrm{Four}(a,b,c,b,a,x) = 0$, then $x \neq 0$ and $(a \cdot y^{\mathbf{2}} + b \cdot y + c) - 2 \cdot a = 0$.

(3) Let a, b, c, x, y be real numbers. Suppose $a \neq 0$ and $(b^{\mathbf{2}} - 4 \cdot a \cdot c) + 8 \cdot a^{\mathbf{2}} > 0$ and $y = x + \frac{1}{x}$. Suppose $\mathrm{Four}(a,b,c,b,a,x) = 0$. Let y_1, y_2 be real numbers. Suppose $y_1 = \frac{-b+\sqrt{(b^{\mathbf{2}}-4 \cdot a \cdot c)+8 \cdot a^{\mathbf{2}}}}{2 \cdot a}$ and $y_2 = \frac{-b-\sqrt{(b^{\mathbf{2}}-4 \cdot a \cdot c)+8 \cdot a^{\mathbf{2}}}}{2 \cdot a}$. Then $x \neq 0$ but $x = \frac{y_1+\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2+\sqrt{\Delta(1,-y_2,1)}}{2}$ or $x = \frac{y_1-\sqrt{\Delta(1,-y_1,1)}}{2}$ or $x = \frac{y_2-\sqrt{\Delta(1,-y_2,1)}}{2}$.

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(4) For every real number x holds $x^3 = x^{\mathbf{2}} \cdot x$ and $x^3 \cdot x = x^4$ and $x^{\mathbf{2}} \cdot x^{\mathbf{2}} = x^4$.

(5) For all real numbers x, y such that $x + y \neq 0$ holds $(x+y)^4 = (x^3 + (3 \cdot y \cdot x^{\mathbf{2}} + 3 \cdot y^{\mathbf{2}} \cdot x) + y^3) \cdot x + (x^3 + (3 \cdot y \cdot x^{\mathbf{2}} + 3 \cdot y^{\mathbf{2}} \cdot x) + y^3) \cdot y$.

(6) For all real numbers x, y such that $x + y \neq 0$ holds $(x+y)^4 = x^4 + (4 \cdot y \cdot x^3 + 6 \cdot y^{\mathbf{2}} \cdot x^{\mathbf{2}} + 4 \cdot y^3 \cdot x) + y^4$.

(7) Let a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 be real numbers. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four}(b_1, b_2, b_3, b_4, b_5, x)$. Then $a_5 = b_5$ and $((a_1 - a_2) + a_3) - a_4 = ((b_1 - b_2) + b_3) - b_4$ and $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$.

(8) Let a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 be real numbers. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four}(b_1, b_2, b_3, b_4, b_5, x)$. Then $a_1 - b_1 = b_3 - a_3$ and $a_2 - b_2 = b_4 - a_4$.

(9) Let a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 be real numbers. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four}(b_1, b_2, b_3, b_4, b_5, x)$. Then $a_1 = b_1$ and $a_2 = b_2$ and $a_3 = b_3$ and $a_4 = b_4$ and $a_5 = b_5$.

Let $a_1, x_1, x_2, x_3, x_4, x$ be real numbers. The functor $\mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$ is defined by:

(Def. 2) $\mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x) = a_1 \cdot ((x - x_1) \cdot (x - x_2) \cdot (x - x_3) \cdot (x - x_4))$.

Let a_1, x_1, x_2, x_3, x_4, x be real numbers.

One can verify that $\mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$ is real.

One can prove the following propositions:

(10) Let a_1, a_2, a_3, a_4, a_5, x, x_1, x_2, x_3, x_4 be real numbers. Suppose $a_1 \neq 0$. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$. Then $\frac{a_1 \cdot x^4 + a_2 \cdot x^3 + a_3 \cdot x^{\mathbf{2}} + a_4 \cdot x + a_5}{a_1} = ((x^{\mathbf{2}} \cdot x^{\mathbf{2}} - (x_1 + x_2 + x_3) \cdot (x^{\mathbf{2}} \cdot x)) + (x_1 \cdot x_3 + x_2 \cdot x_3 + x_1 \cdot x_2) \cdot x^{\mathbf{2}}) - x_1 \cdot x_2 \cdot x_3 \cdot x - (x - x_1) \cdot (x - x_2) \cdot (x - x_3) \cdot x_4$.

(11) Let a_1, a_2, a_3, a_4, a_5, x, x_1, x_2, x_3, x_4 be real numbers. Suppose $a_1 \neq 0$. Suppose that for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$. Then $\frac{a_1 \cdot x^4 + a_2 \cdot x^3 + a_3 \cdot x^{\mathbf{2}} + a_4 \cdot x + a_5}{a_1} = (((x^4 - (x_1 + x_2 + x_3 + x_4) \cdot x^3) + (x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + (x_2 \cdot x_3 + x_2 \cdot x_4) + x_3 \cdot x_4) \cdot x^{\mathbf{2}}) - (x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4) \cdot x) + x_1 \cdot x_2 \cdot x_3 \cdot x_4$.

(12) Let a_1, a_2, a_3, a_4, a_5, x_1, x_2, x_3, x_4 be real numbers. Suppose $a_1 \neq 0$ and for every real number x holds $\mathrm{Four}(a_1, a_2, a_3, a_4, a_5, x) = \mathrm{Four0}(a_1, x_1, x_2, x_3, x_4, x)$. Then $\frac{a_2}{a_1} = -(x_1 + x_2 + x_3 + x_4)$ and $\frac{a_3}{a_1} = x_1 \cdot x_2 + x_1 \cdot x_3 + x_1 \cdot x_4 + (x_2 \cdot x_3 + x_2 \cdot x_4) + x_3 \cdot x_4$ and $\frac{a_4}{a_1} = -(x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_4 + x_1 \cdot x_3 \cdot x_4 + x_2 \cdot x_3 \cdot x_4)$ and $\frac{a_5}{a_1} = x_1 \cdot x_2 \cdot x_3 \cdot x_4$.

(13) Let a, k, y be real numbers. Suppose $a \neq 0$. Suppose that for every real number x holds $x^4 + a^4 = k \cdot a \cdot x \cdot (x^{\mathbf{2}} + a^{\mathbf{2}})$. Then $(y^4 - k \cdot y^3 - k \cdot y) + 1 = 0$.

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