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Improvement of Radix-2^k Signed-Digit Number for High Speed Circuit

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Summary. In this article, a new radix-2^k signed-digit number (Radix-2^k sub signed-digit number) is defined and its properties for hardware realization are discussed.

Until now, high speed calculation method with Radix-2^k signed-digit numbers is proposed, but this method used "Compares With 2" to calculate carry. "Compares with 2" is a very simple method, but it needs very complicated hardware especially when the value of k becomes large. In this article, we propose a subset of Radix-2^k signed-digit, named Radix-2^k sub signed-digit numbers. Radix-2^k sub signed-digit was designed so that the carry calculation use "bit compare" to hardware-realization simplifies more.

In the first section of this article, we defined the concept of Radix-2^k sub signed-digit numbers and proved some of their properties. In the second section, we defined the new carry calculation method in consideration of hardware-realization, and proved some of their properties. In the third section, we provide some functions for generating Radix-2^k sub signed-digit numbers from Radix-2^k signed-digit numbers. In the last section, we defined some functions for generation natural numbers from Radix-2^k sub signed-digit, and we clarified its correctness.

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The articles [11], [14], [8], [12], [1], [4], [3], [13], [10], [7], [2], [9], [5], and [6] provide the notation and terminology for this paper.

1. Definition for Radix-2^k Sub Signed-Digit Number

We adopt the following convention: i, n, m, k, x are natural numbers and i_1, i_2 are integers.

Next we state the proposition

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(1) $((\mathrm{Radix}\,k)^{n}_{\mathbb{N}}) \cdot \mathrm{Radix}\,k = (\mathrm{Radix}\,k)^{n+1}_{\mathbb{N}}$.

Let us consider k. The functor k –SD_Sub_S is defined as follows:

(Def. 1) k –SD_Sub_S $= \{e; e$ ranges over elements of $\mathbb{Z}$: $e \geqslant -\mathrm{Radix}(k -' 1) \wedge e \leqslant \mathrm{Radix}(k -' 1) - 1\}$.

Let us consider k. The functor k –SD_Sub is defined by:

(Def. 2) k –SD_Sub $= \{e; e$ ranges over elements of $\mathbb{Z}$: $e \geqslant -\mathrm{Radix}(k -' 1) - 1 \wedge e \leqslant \mathrm{Radix}(k -' 1)\}$.

The following propositions are true:

(2) If $i_1 \in k$ –SD_Sub, then $-\mathrm{Radix}(k -' 1) - 1 \leqslant i_1$ and $i_1 \leqslant \mathrm{Radix}(k -' 1)$.

(3) For every natural number k holds k –SD_Sub_S $\subseteq k$ –SD_Sub .

(4) k –SD_Sub_S $\subseteq (k+1)$ –SD_Sub_S .

(5) For every natural number k such that $2 \leqslant k$ holds k –SD_Sub $\subseteq k$ –SD .

(6) $0 \in 0$ –SD_Sub_S .

(7) $0 \in k$ –SD_Sub_S .

(8) $0 \in k$ –SD_Sub .

(9) For every set e such that $e \in k$ –SD_Sub holds e is an integer.

(10) k –SD_Sub $\subseteq \mathbb{Z}$.

(11) k –SD_Sub_S $\subseteq \mathbb{Z}$.

Let us consider k. One can verify that k –SD_Sub_S is non empty.

Let us consider k. Note that k –SD_Sub is non empty.

Let us consider k. Then k –SD_Sub_S is a non empty subset of $\mathbb{Z}$.

Let us consider k. Then k –SD_Sub is a non empty subset of $\mathbb{Z}$.

In the sequel a denotes a n-tuple of k –SD and a_1 denotes a n-tuple of k –SD_Sub.

One can prove the following proposition

(12) If $i \in \mathrm{Seg}\,n$, then $a_1(i)$ is an element of k –SD_Sub.

2. Definition for New Carry Calculation Method

Let x be an integer and let k be a natural number.

The functor SDSubAddCarry(x, k) yields an integer and is defined as follows:

(Def. 3) $\mathrm{SDSubAddCarry}(x, k) = \begin{cases} 1, & \text{if } \mathrm{Radix}(k -' 1) \leqslant x, \\ -1, & \text{if } x < -\mathrm{Radix}(k -' 1), \\ 0, & \text{otherwise.} \end{cases}$

Let x be an integer and let k be a natural number.

The functor SDSubAddData(x, k) yields an integer and is defined as follows:

(Def. 4) $\mathrm{SDSubAddData}(x, k) = x - \mathrm{Radix}\,k \cdot \mathrm{SDSubAddCarry}(x, k)$.

One can prove the following propositions:

(13) For every integer x and for every natural number k such that $2 \leqslant k$ holds $-1 \leqslant \mathrm{SDSubAddCarry}(x, k)$ and $\mathrm{SDSubAddCarry}(x, k) \leqslant 1$.

(14) If $2 \leqslant k$ and $i_1 \in k$ –SD, then $\mathrm{SDSubAddData}(i_1, k) \geqslant -\mathrm{Radix}(k -' 1)$ and $\mathrm{SDSubAddData}(i_1, k) \leqslant \mathrm{Radix}(k -' 1) - 1$.

(15) For every integer x and for every natural number k such that $2 \leqslant k$ holds $\mathrm{SDSubAddCarry}(x, k) \in k$ –SD_Sub_S.

(16) If $2 \leqslant k$ and $i_1 \in k$ –SD and $i_2 \in k$ –SD, then $\mathrm{SDSubAddData}(i_1, k) + \mathrm{SDSubAddCarry}(i_2, k) \in k$ –SD_Sub.

(17) If $2 \leqslant k$, then $\mathrm{SDSubAddCarry}(0, k) = 0$.

3. Definition for Translation from Radix-2^k Signed-Digit Number

Let i, k, n be natural numbers and let x be a n-tuple of k –SD_Sub. The functor DigA_SDSub(x, i) yields an integer and is defined as follows:

(Def. 5)(i) DigA_SDSub$(x, i) = x(i)$ if $i \in \mathrm{Seg}\, n$,

(ii) DigA_SDSub$(x, i) = 0$ if $i = 0$.

Let i, k, n be natural numbers and let x be a n-tuple of k –SD. The functor SD2SDSubDigit(x, i, k) yields an integer and is defined by:

$$\text{(Def. 6)} \quad \mathrm{SD2SDSubDigit}(x, i, k) = \begin{cases} \text{(i)}\ \mathrm{SDSubAddData}(\mathrm{DigA}(x, i), k) + \\ \quad \mathrm{SDSubAddCarry}(\mathrm{DigA}(x, i -' 1), k), \\ \quad \text{if } i \in \mathrm{Seg}\, n, \\ \text{(ii)}\ \mathrm{SDSubAddCarry}(\mathrm{DigA}(x, i -' 1), k), \\ \quad \text{if } i = n + 1, \\ 0, \text{ otherwise.} \end{cases}$$

We now state the proposition

(18) If $2 \leqslant k$ and $i \in \mathrm{Seg}(n + 1)$, then SD2SDSubDigit$(a, i, k)$ is an element of k –SD_Sub.

Let i, k, n be natural numbers and let x be a n-tuple of k –SD. Let us assume that $2 \leqslant k$ and $i \in \mathrm{Seg}(n + 1)$. The functor SD2SDSubDigitS(x, i, k) yielding an element of k –SD_Sub is defined by:

(Def. 7) SD2SDSubDigitS(x, i, k) = SD2SDSubDigit(x, i, k).

Let n, k be natural numbers and let x be a n-tuple of k –SD. The functor SD2SDSub x yielding a $n + 1$-tuple of k –SD_Sub is defined by:

(Def. 8) For every natural number i such that $i \in \mathrm{Seg}(n + 1)$ holds DigA_SDSub(SD2SDSub x, i) = SD2SDSubDigitS(x, i, k).

Next we state two propositions:

(19) If $i \in \mathrm{Seg}\, n$, then DigA_SDSub(a_1, i) is an element of k –SD_Sub.

(20) If $2 \leqslant k$ and $i_1 \in k$ –SD and $i_2 \in k$ –SD_Sub, then $\mathrm{SDSubAddData}(i_1 + i_2, k) \in k$ –SD_Sub_S.

4. Definiton for Translation from Radix-2^k Sub Signed-Digit Number to INT

Let i, k, n be natural numbers and let x be a n-tuple of k −SD_Sub. The functor DigB_SDSub(x, i) yielding an element of $\mathbb{Z}$ is defined by:

(Def. 9) DigB_SDSub(x, i) = DigA_SDSub(x, i).

Let i, k, n be natural numbers and let x be a n-tuple of k −SD_Sub. The functor SDSub2INTDigit(x, i, k) yielding an element of $\mathbb{Z}$ is defined as follows:

(Def. 10) SDSub2INTDigit$(x, i, k) = ((\mathrm{Radix}\, k)_{\mathbb{N}}^{i-'1}) \cdot$ DigB_SDSub(x, i).

Let n, k be natural numbers and let x be a n-tuple of k −SD_Sub. The functor SDSub2INT x yields a n-tuple of $\mathbb{Z}$ and is defined as follows:

(Def. 11) For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $(\mathrm{SDSub2INT}\, x)_i =$ SDSub2INTDigit(x, i, k).

Let n, k be natural numbers and let x be a n-tuple of k −SD_Sub. The functor SDSub2IntOut x yields an integer and is defined as follows:

(Def. 12) SDSub2IntOut $x = \sum$ SDSub2INT x.

Next we state two propositions:

(21) For every i such that $i \in \mathrm{Seg}\, n$ holds if $2 \leqslant k$, then DigA_SDSub(SD2SDSub DecSD$(m, n+1, k), i)$ = DigA_SDSub(SD2SDSub DecSD$(m \bmod (\mathrm{Radix}\, k)_{\mathbb{N}}^{n}, n, k), i)$.

(22) For every n such that $n \geqslant 1$ and for all k, x such that $k \geqslant 2$ and x is represented by n, k holds x = SDSub2IntOut SD2SDSub DecSD(x, n, k).

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