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On Some Properties of Real Hilbert Space. Part II

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Summary. This paper is a continuation of our paper [21]. We give an analogue of the necessary and sufficient condition for summable set (i.e. the main theorem of [21]) with respect to summable set by a functional L in real Hilbert space. After presenting certain useful lemmas, we prove our main theorem that the summability for an orthonormal infinite set in real Hilbert space is equivalent to its summability with respect to the square of norm, say $H(x) = (x, x)$. Then we show that the square of norm H commutes with infinite sum operation if the orthonormal set under our consideration is summable. Our main theorem is due to [7].

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The articles [16], [18], [5], [14], [8], [3], [4], [19], [17], [11], [12], [13], [2], [6], [9], [15], [10], [1], [20], and [21] provide the notation and terminology for this paper.

1. Necessary and Sufficient Condition for Summable Set

In this paper X is a real unitary space and x, y are points of X.

The following propositions are true:

(1) Let Y be a subset of the carrier of X and L be a functional in X. Then Y is summable set by L if and only if for every real number e such that $0 < e$ there exists a finite subset Y_0 of the carrier of X such that Y_0 is non

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empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of the carrier of X such that Y_1 is non empty and $Y_1 \subseteq Y$ and Y_0 misses Y_1 holds $|\mathrm{setopfunc}(Y_1, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})| < e$.

(2) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let S be a finite orthogonal family of X. Suppose S is non empty. Let I be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \mathrm{dom}\, I$ and for every y such that $y \in S$ holds $I(y) = y$. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \mathrm{dom}\, H$ and for every y such that $y \in S$ holds $H(y) = (y|y)$. Then $(\mathrm{setopfunc}(S, \text{the carrier of } X, \text{the carrier of } X, I, \text{the addition of } X)|\, \mathrm{setopfunc}(S, \text{the carrier of } X, \text{the carrier of } X, I, \text{the addition of } X)) = \mathrm{setopfunc}(S, \text{the carrier of } X, \mathbb{R}, H, +_{\mathbb{R}})$.

(3) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let S be a finite orthogonal family of X. Suppose S is non empty. Let H be a functional in X. Suppose $S \subseteq \mathrm{dom}\, H$ and for every x such that $x \in S$ holds $H(x) = (x|x)$. Then $(\mathrm{Setsum}(S)|\, \mathrm{Setsum}(S)) = \mathrm{setopfunc}(S, \text{the carrier of } X, \mathbb{R}, H, +_{\mathbb{R}})$.

(4) Let Y be an orthogonal family of X and Z be a subset of the carrier of X. If Z is a subset of Y, then Z is an orthogonal family of X.

(5) Let Y be an orthonormal family of X and Z be a subset of the carrier of X. If Z is a subset of Y, then Z is an orthonormal family of X.

2. Equivalence of Summability

Next we state three propositions:

(6) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity and X is a Hilbert space. Let S be an orthonormal family of X and H be a functional in X. Suppose $S \subseteq \mathrm{dom}\, H$ and for every x such that $x \in S$ holds $H(x) = (x|x)$. Then S is summable_set if and only if S is summable set by H.

(7) Let S be a subset of the carrier of X. Suppose S is non empty and summable_set. Let e be a real number. Suppose $0 < e$. Then there exists a finite subset Y_0 of the carrier of X such that

(i) Y_0 is non empty,

(ii) $Y_0 \subseteq S$, and

(iii) for every finite subset Y_1 of the carrier of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq S$ holds $|(\mathrm{sum}\, S|\, \mathrm{sum}\, S) - (\mathrm{Setsum}(Y_1)|\, \mathrm{Setsum}(Y_1))| < e$.

(8) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity and X is a Hilbert space. Let S be an orthonormal family of X. Suppose S is non empty. Let H be a functional

in X. Suppose $S \subseteq \operatorname{dom} H$ and for every x such that $x \in S$ holds $H(x) = (x|x)$. If S is summable_set, then $(\operatorname{sum} S|\operatorname{sum} S) = \operatorname{SumByfunc}(S, H)$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[7] P. R. Halmos. *Introduction to Hilbert Space*. American Mathematical Society, 1987.
[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[10] Bogdan Nowak and Andrzej Trybulec. Hahn-Banach theorem. *Formalized Mathematics*, 4(**1**):29–34, 1993.
[11] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[12] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(**4**):511–516, 1991.
[13] Jan Popiołek. Introduction to Banach and Hilbert spaces - part III. *Formalized Mathematics*, 2(**4**):523–526, 1991.
[14] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics*.
[15] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[19] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[20] Hiroshi Yamazaki, Yasunari Shidama, and Yatsuka Nakamura. Bessel's inequality. *Formalized Mathematics*, 11(**2**):169–173, 2003.
[21] Hiroshi Yamazaki, Yasumasa Suzuki, Takao Inoué, and Yasunari Shidama. On some properties of real Hilbert space. Part I. *Formalized Mathematics*, 11(**3**):225–229, 2003.

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