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On the Subcontinua of a Real Line[1]

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Summary. In [10] we showed that the only proper subcontinua of the simple closed curve are arcs and single points. In this article we prove that the only proper subcontinua of the real line are closed intervals. We introduce some auxiliary notions such as $]a, b[_{\mathbb{Q}}$, $]a, b[_{\mathbb{IQ}}$ – intervals consisting of rational and irrational numbers respectively. We show also some basic topological properties of intervals.

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The notation and terminology used in this paper are introduced in the following papers: [24], [27], [22], [23], [18], [25], [28], [3], [4], [26], [19], [6], [21], [13], [16], [17], [1], [8], [5], [9], [14], [7], [20], [15], [12], [11], and [2].

1. Preliminaries

The following three propositions are true:

(1) For all sets A, B, C, D holds $(A \cup B \cup C) \cup D = A \cup (B \cup C \cup D)$.

(2) For all sets A, B, a such that $A \subseteq B$ and $B \subseteq A \cup \{a\}$ holds $A \cup \{a\} = B$ or $A = B$.

(3) For all sets x_1, x_2, x_3, x_4, x_5, x_6 holds $\{x_1, x_2, x_3, x_4, x_5, x_6\} = \{x_1, x_3, x_6\} \cup \{x_2, x_4, x_5\}$.

In the sequel x_1, x_2, x_3, x_4, x_5, x_6, x_7 are sets.

Let x_1, x_2, x_3, x_4, x_5, x_6 be sets. We say that x_1, x_2, x_3, x_4, x_5, x_6 are mutually different if and only if the conditions (Def. 1) are satisfied.

(Def. 1) $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_1 \neq x_6$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_2 \neq x_6$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_3 \neq x_6$ and $x_4 \neq x_5$ and $x_4 \neq x_6$ and $x_5 \neq x_6$.

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Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be sets. We say that x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different if and only if the conditions (Def. 2) are satisfied.

(Def. 2) $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_1 \neq x_6$ and $x_1 \neq x_7$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_2 \neq x_6$ and $x_2 \neq x_7$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_3 \neq x_6$ and $x_3 \neq x_7$ and $x_4 \neq x_5$ and $x_4 \neq x_6$ and $x_4 \neq x_7$ and $x_5 \neq x_6$ and $x_5 \neq x_7$ and $x_6 \neq x_7$.

One can prove the following propositions:

(4) For all sets x_1, x_2, x_3, x_4, x_5, x_6 such that x_1, x_2, x_3, x_4, x_5, x_6 are mutually different holds $\mathrm{card}\{x_1, x_2, x_3, x_4, x_5, x_6\} = 6$.

(5) For all sets x_1, x_2, x_3, x_4, x_5, x_6, x_7 such that x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different holds $\mathrm{card}\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} = 7$.

(6) If $\{x_1, x_2, x_3\}$ misses $\{x_4, x_5, x_6\}$, then $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_1 \neq x_6$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_2 \neq x_6$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_3 \neq x_6$.

(7) Suppose x_1, x_2, x_3 are mutually different and x_4, x_5, x_6 are mutually different and $\{x_1, x_2, x_3\}$ misses $\{x_4, x_5, x_6\}$. Then x_1, x_2, x_3, x_4, x_5, x_6 are mutually different.

(8) Suppose x_1, x_2, x_3, x_4, x_5, x_6 are mutually different and $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ misses $\{x_7\}$. Then x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different.

(9) If x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different, then x_7, x_1, x_2, x_3, x_4, x_5, x_6 are mutually different.

(10) If x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different, then x_1, x_2, x_5, x_3, x_6, x_7, x_4 are mutually different.

(11) Let T be a non empty topological space and a, b be points of T. Given a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$. Then there exists a map g from $\mathbb{I}$ into T such that g is continuous and $g(0) = b$ and $g(1) = a$.

Let us observe that $\mathbb{R}^{\mathbf{1}}$ is arcwise connected.

Let us note that there exists a topological space which is connected and non empty.

2. Intervals

The following two propositions are true:

(12) Every subset of $\mathbb{R}$ is a subset of $\mathbb{R}^{\mathbf{1}}$.

(13) $\Omega_{\mathbb{R}^{\mathbf{1}}} = \mathbb{R}$.

Let a be a real number. We introduce $]-\infty, a]$ as a synonym of $]-\infty, a]$. We introduce $]-\infty, a[$ as a synonym of $]-\infty, a[$. We introduce $[a, +\infty[$ as a synonym of $[a, +\infty[$. We introduce $]a, +\infty[$ as a synonym of $]a, +\infty[$.

Next we state a number of propositions:

(14) For all real numbers a, b holds $a \in]b, +\infty[$ iff $a > b$.

(15) For all real numbers a, b holds $a \in [b, +\infty[$ iff $a \geqslant b$.

(16) For all real numbers a, b holds $a \in]-\infty, b]$ iff $a \leqslant b$.

(17) For all real numbers a, b holds $a \in]-\infty, b[$ iff $a < b$.

(18) For every real number a holds $\mathbb{R} \setminus \{a\} =]-\infty, a[\cup]a, +\infty[$.

(19) For all real numbers a, b, c, d such that $a < b$ and $b \leqslant c$ holds $[a, b]$ misses $]c, d]$.

(20) For all real numbers a, b, c, d such that $a < b$ and $b \leqslant c$ holds $[a, b[$ misses $[c, d]$.

(21) Let A, B be subsets of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $b \leqslant c$ and $c < d$ and $A = [a, b[$ and $B =]c, d]$. Then A and B are separated.

(22) For every real number a holds $\mathbb{R} \setminus]-\infty, a[= [a, +\infty[$.

(23) For every real number a holds $\mathbb{R} \setminus]-\infty, a] =]a, +\infty[$.

(24) For every real number a holds $\mathbb{R} \setminus]a, +\infty[=]-\infty, a]$.

(25) For every real number a holds $\mathbb{R} \setminus [a, +\infty[=]-\infty, a[$.

(26) For every real number a holds $]-\infty, a]$ misses $]a, +\infty[$.

(27) For every real number a holds $]-\infty, a[$ misses $[a, +\infty[$.

(28) For all real numbers a, b, c such that $a \leqslant c$ and $c \leqslant b$ holds $[a, b] \cup [c, +\infty[= [a, +\infty[$.

(29) For all real numbers a, b, c such that $a \leqslant c$ and $c \leqslant b$ holds $]-\infty, c] \cup [a, b] =]-\infty, b]$.

(30) For every 1-sorted structure T and for every subset A of T holds $\{A\}$ is a family of subsets of T.

(31) For every 1-sorted structure T and for all subsets A, B of T holds $\{A, B\}$ is a family of subsets of T.

(32) For every 1-sorted structure T and for all subsets A, B, C of T holds $\{A, B, C\}$ is a family of subsets of T.

Let us observe that every element of $\mathbb{Q}$ is real.

Let us observe that every element of the carrier of the metric space of real numbers is real.

Next we state four propositions:

(33) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and p be a point of the metric space of real numbers. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ holds $\mathrm{Ball}(p, r)$ meets A.

(34) For all elements p, q of the carrier of the metric space of real numbers such that $q \geqslant p$ holds $\rho(p, q) = q - p$.

(35) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{Q}$ holds $\overline{A} =$ the carrier of $\mathbb{R}^{\mathbf{1}}$.

(36) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $A =]a, b[$ and $a \neq b$ holds $\overline{A} = [a, b]$.

3. Rational and Irrational Numbers

Let us mention that e is irrational.

The subset $\mathbb{IQ}$ of $\mathbb{R}$ is defined by:

(Def. 3) $\mathbb{IQ} = \mathbb{R} \setminus \mathbb{Q}$.

Let a, b be real numbers. The functor $]a, b[_{\mathbb{Q}}$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 4) $]a, b[_{\mathbb{Q}} = \mathbb{Q} \cap]a, b[$.

The functor $]a, b[_{\mathbb{IQ}}$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 5) $]a, b[_{\mathbb{IQ}} = \mathbb{IQ} \cap]a, b[$.

One can prove the following proposition

(37) For every real number x holds x is irrational iff $x \in \mathbb{IQ}$.

Let us observe that there exists a real number which is irrational.

Let us note that $\mathbb{IQ}$ is non empty.

Next we state several propositions:

(38) For every rational number a and for every irrational real number b holds $a + b$ is irrational.

(39) For every irrational real number a holds $-a$ is irrational.

(40) For every rational number a and for every irrational real number b holds $a - b$ is irrational.

(41) For every rational number a and for every irrational real number b holds $b - a$ is irrational.

(42) For every rational number a and for every irrational real number b such that $a \neq 0$ holds $a \cdot b$ is irrational.

(43) For every rational number a and for every irrational real number b such that $a \neq 0$ holds $\frac{b}{a}$ is irrational.

One can check that every real number which is irrational is also non zero.

The following propositions are true:

(44) For every rational number a and for every irrational real number b such that $a \neq 0$ holds $\frac{a}{b}$ is irrational.

(45) For every irrational real number r holds $\mathrm{frac}\, r$ is irrational.

Let r be an irrational real number. Note that $\mathrm{frac}\, r$ is irrational.

Let a be an irrational real number. Note that $-a$ is irrational.

Let a be a rational number and let b be an irrational real number. One can verify the following observations:

* $a+b$ is irrational,
* $b+a$ is irrational,
* $a-b$ is irrational, and
* $b-a$ is irrational.

Let us observe that there exists a rational number which is non zero.

Let a be a non zero rational number and let b be an irrational real number. One can check the following observations:

* $a \cdot b$ is irrational,
* $b \cdot a$ is irrational,
* $\frac{a}{b}$ is irrational, and
* $\frac{b}{a}$ is irrational.

The following propositions are true:

(46) For every irrational real number r holds $0 < \operatorname{frac} r$.

(47) For all real numbers a, b such that $a < b$ there exist rational numbers p_1, p_2 such that $a < p_1$ and $p_1 < p_2$ and $p_2 < b$.

(48) For all real numbers s_1, s_3, s_4, l such that $s_1 \leqslant s_3$ and $s_1 < s_4$ and $0 < l$ and $l < 1$ holds $s_1 < (1-l) \cdot s_3 + l \cdot s_4$.

(49) For all real numbers s_1, s_3, s_4, l such that $s_3 < s_1$ and $s_4 \leqslant s_1$ and $0 < l$ and $l < 1$ holds $(1-l) \cdot s_3 + l \cdot s_4 < s_1$.

(50) For all real numbers a, b such that $a < b$ there exists an irrational real number p such that $a < p$ and $p < b$.

(51) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{IQ}$ holds $\overline{A} =$ the carrier of $\mathbb{R}^{\mathbf{1}}$.

(52) For all real numbers a, b, c such that $a < b$ holds $c \in]a,b[_{\mathbb{Q}}$ iff c is rational and $a < c$ and $c < b$.

(53) For all real numbers a, b, c such that $a < b$ holds $c \in]a,b[_{\mathbb{IQ}}$ iff c is irrational and $a < c$ and $c < b$.

(54) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]a,b[_{\mathbb{Q}}$ holds $\overline{A} = [a,b]$.

(55) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]a,b[_{\mathbb{IQ}}$ holds $\overline{A} = [a,b]$.

(56) For every connected topological space T and for every closed open subset A of T holds $A = \emptyset$ or $A = \Omega_T$.

(57) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that A is closed and open holds $A = \emptyset$ or $A = \mathbb{R}$.

4. Topological Properties of Intervals

We now state a number of propositions:

(58) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $A = [a, b[$ and $a \neq b$ holds $\overline{A} = [a, b]$.

(59) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $A =]a, b]$ and $a \neq b$ holds $\overline{A} = [a, b]$.

(60) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c be real numbers. If $A = [a, b[\cup]b, c]$ and $a < b$ and $b < c$, then $\overline{A} = [a, c]$.

(61) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = \{a\}$ holds $\overline{A} = \{a\}$.

(62) For every subset A of $\mathbb{R}$ and for every subset B of $\mathbb{R}^{\mathbf{1}}$ such that $A = B$ holds A is open iff B is open.

(63) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]a, +\infty[$ holds A is open.

(64) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty, a[$ holds A is open.

(65) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty, a]$ holds A is closed.

(66) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = [a, +\infty[$ holds A is closed.

(67) For every real number a holds $[a, +\infty[= \{a\}\cup]a, +\infty[$.

(68) For every real number a holds $]-\infty, a] = \{a\}\cup]-\infty, a[$.

(69) For every real number a holds $]a, +\infty[\subseteq [a, +\infty[$.

(70) For every real number a holds $]-\infty, a[\subseteq]-\infty, a]$.

Let a be a real number. One can check the following observations:

* $]a, +\infty[$ is non empty,
* $]-\infty, a]$ is non empty,
* $]-\infty, a[$ is non empty, and
* $[a, +\infty[$ is non empty.

The following propositions are true:

(71) For every real number a holds $]a, +\infty[\neq \mathbb{R}$.

(72) For every real number a holds $[a, +\infty[\neq \mathbb{R}$.

(73) For every real number a holds $]-\infty, a] \neq \mathbb{R}$.

(74) For every real number a holds $]-\infty, a[\neq \mathbb{R}$.

(75) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]a, +\infty[$ holds $\overline{A} = [a, +\infty[$.

(76) For every real number a holds $\overline{]a, +\infty[} = [a, +\infty[$.

(77) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty, a[$ holds $\overline{A} =]-\infty, a]$.

(78) For every real number a holds $\overline{]-\infty, a[} =]-\infty, a]$.

(79) Let A, B be subsets of the carrier of $\mathbb{R}^{\mathbf{1}}$ and b be a real number. If $A =]-\infty, b[$ and $B =]b, +\infty[$, then A and B are separated.

(80) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A = [a, b[\cup]b, +\infty[$ holds $\overline{A} = [a, +\infty[$.

(81) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]a, b[\cup]b, +\infty[$ holds $\overline{A} = [a, +\infty[$.

(82) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c be real numbers. If $a < b$ and $b < c$ and $A =]a, b[_{\mathbb{Q}} \cup]b, c[\cup]c, +\infty[$, then $\overline{A} = [a, +\infty[$.

(83) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ holds $-A = \mathbb{R} \setminus A$.

(84) For all real numbers a, b such that $a < b$ holds $]a, b[_{\mathbb{IQ}}$ misses $]a, b[_{\mathbb{Q}}$.

(85) For all real numbers a, b such that $a < b$ holds $\mathbb{R} \setminus]a, b[_{\mathbb{Q}} =]-\infty, a] \cup]a, b[_{\mathbb{IQ}} \cup [b, +\infty[$.

(86) For all real numbers a, b, c such that $a \leqslant b$ and $b < c$ holds $a \notin]b, c[\cup]c, +\infty[$.

(87) For all real numbers a, b such that $a < b$ holds $b \notin]a, b[\cup]b, +\infty[$.

(88) For all real numbers a, b such that $a < b$ holds $[a, +\infty[\setminus (]a, b[\cup]b, +\infty[) = \{a\} \cup \{b\}$.

(89) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that $A =]2, 3[_{\mathbb{Q}} \cup]3, 4[\cup]4, +\infty[$ holds $-A =]-\infty, 2] \cup]2, 3[_{\mathbb{IQ}} \cup \{3\} \cup \{4\}$.

(90) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = \{a\}$ holds $-A =]-\infty, a[\cup]a, +\infty[$.

(91) For all real numbers a, b such that $a < b$ holds $]a, +\infty[\cap]-\infty, b] =]a, b]$.

(92) $(]-\infty, 1[\cup]1, +\infty[) \cap (]-\infty, 2] \cup]2, 3[_{\mathbb{IQ}} \cup \{3\} \cup \{4\}) =]-\infty, 1[\cup]1, 2] \cup]2, 3[_{\mathbb{IQ}} \cup \{3\} \cup \{4\}$.

(93) For all real numbers a, b such that $a \leqslant b$ holds $]-\infty, b[\setminus \{a\} =]-\infty, a[\cup]a, b[$.

(94) For all real numbers a, b such that $a \leqslant b$ holds $]a, +\infty[\setminus \{b\} =]a, b[\cup]b, +\infty[$.

(95) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b be real numbers. If $a \leqslant b$ and $A = \{a\} \cup [b, +\infty[$, then $-A =]-\infty, a[\cup]a, b[$.

(96) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]-\infty, a[\cup]a, b[$ holds $\overline{A} =]-\infty, b]$.

(97) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]-\infty, a[\cup]a, b]$ holds $\overline{A} =]-\infty, b]$.

(98) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty, a]$ holds $-A =]a, +\infty[$.

(99) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = [a, +\infty[$ holds $-A =]-\infty, a[$.

(100) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c be real numbers. If $a < b$ and $b < c$ and $A =]-\infty, a[\cup]a, b] \cup]b, c[_{\mathbb{IQ}} \cup \{c\}$, then $\overline{A} =]-\infty, c]$.

(101) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b, c, d be real numbers. If $a < b$ and $b < c$ and $A =]-\infty, a[\cup]a, b] \cup]b, c[_{\mathbb{IQ}} \cup \{c\} \cup \{d\}$, then $\overline{A} =]-\infty, c] \cup \{d\}$.

(102) Let A be a subset of the carrier of $\mathbb{R}^{\mathbf{1}}$ and a, b be real numbers. If $a \leqslant b$ and $A =]-\infty, a] \cup \{b\}$, then $-A =]a, b[\cup]b, +\infty[$.

(103) For all real numbers a, b holds $[a, +\infty[\cup \{b\} \neq \mathbb{R}$.

(104) For all real numbers a, b holds $]-\infty, a] \cup \{b\} \neq \mathbb{R}$.

(105) For every topological structure T_1 and for all subsets A, B of the carrier of T_1 such that $A \neq B$ holds $-A \neq -B$.

(106) For every subset A of the carrier of $\mathbb{R}^{\mathbf{1}}$ such that $\mathbb{R} = -A$ holds $A = \emptyset$.

5. Subcontinua of a Real Line

Let us mention that $\mathbb{I}$ is arcwise connected.

We now state several propositions:

(107) Let X be a compact subset of $\mathbb{R}^{\mathbf{1}}$ and X' be a subset of $\mathbb{R}$. If $X' = X$, then X' is upper bounded and lower bounded.

(108) Let X be a compact subset of $\mathbb{R}^{\mathbf{1}}$, X' be a subset of $\mathbb{R}$, and x be a real number. If $x \in X'$ and $X' = X$, then $\inf X' \leqslant x$ and $x \leqslant \sup X'$.

(109) Let C be a non empty compact connected subset of $\mathbb{R}^{\mathbf{1}}$ and C' be a subset of $\mathbb{R}$. If $C = C'$ and $[\inf C', \sup C'] \subseteq C'$, then $[\inf C', \sup C'] = C'$.

(110) Let A be a connected subset of $\mathbb{R}^{\mathbf{1}}$ and a, b, c be real numbers. If $a \leqslant b$ and $b \leqslant c$ and $a \in A$ and $c \in A$, then $b \in A$.

(111) For every connected subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a \in A$ and $b \in A$ holds $[a, b] \subseteq A$.

(112) Every non empty compact connected subset of $\mathbb{R}^{\mathbf{1}}$ is a non empty closed-interval subset of $\mathbb{R}$.

(113) For every non empty compact connected subset A of $\mathbb{R}^{\mathbf{1}}$ there exist real numbers a, b such that $a \leqslant b$ and $A = [a, b]$.

6. Sets with Proper Subsets Only

Let T_1 be a topological structure and let F be a family of subsets of T_1. We say that F has proper subsets if and only if:

(Def. 6) The carrier of $T_1 \notin F$.

One can prove the following proposition

(114) Let T_1 be a topological structure and F, G be families of subsets of T_1 such that F has proper subsets and $G \subseteq F$. Then G has proper subsets.

Let T_1 be a non empty topological structure. Observe that there exists a family of subsets of T_1 which has proper subsets.

We now state the proposition

(115) Let T_1 be a non empty topological structure and A, B be families of subsets of T_1 with proper subsets. Then $A \cup B$ has proper subsets.

Let T be a topological structure and let F be a family of subsets of T. We say that F is open if and only if:

(Def. 7) For every subset P of T such that $P \in F$ holds P is open.

We say that F is closed if and only if:

(Def. 8) For every subset P of T such that $P \in F$ holds P is closed.

Let T be a topological space. Note that there exists a family of subsets of T which is open, closed, and non empty.

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