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Full Subtracter Circuit. Part II

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Summary. In this article we continue investigations from [22] of verification of a design of subtracter circuit. We define it as a combination of multi cell circuit using schemes from [6]. As the main result we prove the stability of the circuit.

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The articles [17], [16], [21], [15], [3], [18], [25], [1], [9], [10], [4], [8], [2], [19], [24], [14], [20], [13], [12], [11], [23], [5], [7], and [22] provide the terminology and notation for this paper.

Let n be a natural number and let x, y be finite sequences. The functor n-BitSubtracterStr(x, y) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by the condition (Def. 1).

(Def. 1) There exist many sorted sets f, g indexed by $\mathbb{N}$ such that
 (i) n-BitSubtracterStr$(x, y) = f(n)$,
 (ii) $f(0) = $ 1GateCircStr$(\varepsilon, Boolean^0 \longmapsto true)$,
 (iii) $g(0) = \langle \varepsilon, Boolean^0 \longmapsto true \rangle$, and
 (iv) for every natural number n and for every non empty many sorted signature S and for every set z such that $S = f(n)$ and $z = g(n)$ holds $f(n+1) = S +\cdot$ BitSubtracterWithBorrowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = $ BorrowOutput$(x(n+1), y(n+1), z)$.

Let n be a natural number and let x, y be finite sequences. The functor n-BitSubtracterCirc(x, y) yielding a Boolean strict circuit of

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n-BitSubtracterStr(x, y) with denotation held in gates is defined by the condition (Def. 2).

(Def. 2) There exist many sorted sets f, g, h indexed by $\mathbb{N}$ such that

(i) n-BitSubtracterStr$(x, y) = f(n)$,

(ii) n-BitSubtracterCirc$(x, y) = g(n)$,

(iii) $f(0) =$ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(iv) $g(0) =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(v) $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$, and

(vi) for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitSubtracterWithBorrowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot$ BitSubtracterWithBorrowCirc$(x(n+1), y(n+1), z)$ and $h(n+1) =$ BorrowOutput$(x(n+1), y(n+1), z)$.

Let n be a natural number and let x, y be finite sequences. The functor n-BitBorrowOutput(x, y) yields an element of InnerVertices(n-BitSubtracterStr (x, y)) and is defined by the condition (Def. 3).

(Def. 3) There exists a many sorted set h indexed by $\mathbb{N}$ such that

(i) n-BitBorrowOutput$(x, y) = h(n)$,

(ii) $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$, and

(iii) for every natural number n and for every set z such that $z = h(n)$ holds $h(n+1) =$ BorrowOutput$(x(n+1), y(n+1), z)$.

One can prove the following propositions:

(1) Let x, y be finite sequences and f, g, h be many sorted sets indexed by $\mathbb{N}$. Suppose that

(i) $f(0) =$ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(ii) $g(0) =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$,

(iii) $h(0) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$, and

(iv) for every natural number n and for every non empty many sorted signature S and for every non-empty algebra A over S and for every set z such that $S = f(n)$ and $A = g(n)$ and $z = h(n)$ holds $f(n+1) = S +\cdot$ BitSubtracterWithBorrowStr$(x(n+1), y(n+1), z)$ and $g(n+1) = A +\cdot$ BitSubtracterWithBorrowCirc$(x(n+1), y(n+1), z)$ and $h(n+1) =$ BorrowOutput$(x(n+1), y(n+1), z)$.

Let n be a natural number. Then n-BitSubtracterStr$(x, y) = f(n)$ and n-BitSubtracterCirc$(x, y) = g(n)$ and n-BitBorrowOutput$(x, y) = h(n)$.

(2) For all finite sequences a, b holds 0-BitSubtracterStr$(a, b) =$ 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$ and 0-BitSubtracterCirc$(a, b) =$ 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$ and 0-BitBorrowOutput$(a, b) = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$.

(3) Let a, b be finite sequences and c be a set. Suppose $c = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$. Then 1-BitSubtracterStr(a, b) = 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$+· BitSubtracterWithBorrowStr$(a(1), b(1), c)$ and 1-BitSubtracterCirc(a, b) = 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$+· BitSubtracterWithBorrowCirc$(a(1), b(1), c)$ and 1-BitBorrowOutput(a, b) = BorrowOutput$(a(1), b(1), c)$.

(4) For all sets a, b, c such that $c = \langle \varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true} \rangle$ holds 1-BitSubtracterStr$(\langle a \rangle, \langle b \rangle)$ = 1GateCircStr$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$+· BitSubtracterWithBorrowStr(a, b, c) and 1-BitSubtracterCirc$(\langle a \rangle, \langle b \rangle)$ = 1GateCircuit$(\varepsilon, \mathit{Boolean}^0 \longmapsto \mathit{true})$+· BitSubtracterWithBorrowCirc(a, b, c) and 1-BitBorrowOutput$(\langle a \rangle, \langle b \rangle)$ = BorrowOutput(a, b, c).

(5) Let n be a natural number, p, q be finite sequences with length n, and p_1, p_2, q_1, q_2 be finite sequences. Then n-BitSubtracterStr$(p ^\frown p_1, q ^\frown q_1)$ = n-BitSubtracterStr$(p ^\frown p_2, q ^\frown q_2)$ and n-BitSubtracterCirc$(p ^\frown p_1, q ^\frown q_1)$ = n-BitSubtracterCirc$(p ^\frown p_2, q ^\frown q_2)$ and n-BitBorrowOutput$(p ^\frown p_1, q ^\frown q_1)$ = n-BitBorrowOutput$(p ^\frown p_2, q ^\frown q_2)$.

(6) Let n be a natural number, x, y be finite sequences with length n, and a, b be sets.
Then $(n+1)$-BitSubtracterStr$(x ^\frown \langle a \rangle, y ^\frown \langle b \rangle)$ = (n-BitSubtracterStr(x, y))+· BitSubtracterWithBorrowStr$(a, b, n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitSubtracterCirc$(x ^\frown \langle a \rangle, y ^\frown \langle b \rangle)$ = (n-BitSubtracterCirc(x, y))+· BitSubtracterWithBorrowCirc$(a, b, n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitBorrowOutput$(x ^\frown \langle a \rangle, y ^\frown \langle b \rangle)$ = BorrowOutput$(a, b, n$-BitBorrowOutput$(x, y))$.

(7) Let n be a natural number and x, y be finite sequences. Then $(n+1)$-BitSubtracterStr(x, y) = (n-BitSubtracterStr(x, y))+· BitSubtracterWithBorrowStr$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitSubtracterCirc(x, y) = (n-BitSubtracterCirc(x, y))+· BitSubtracterWithBorrowCirc$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$ and $(n+1)$-BitBorrowOutput(x, y) = BorrowOutput$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$.

(8) For all natural numbers n, m such that $n \leqslant m$ and for all finite sequences x, y holds InnerVertices(n-BitSubtracterStr(x, y)) $\subseteq$ InnerVertices(m-BitSubtracterStr(x, y)).

(9) For every natural number n and for all finite sequences x, y holds InnerVertices($(n+1)$-BitSubtracterStr(x, y)) = InnerVertices(n-BitSubtracterStr(x, y)) $\cup$ InnerVertices (BitSubtracterWithBorrowStr$(x(n+1), y(n+1), n$-BitBorrowOutput$(x, y))$).

Let k, n be natural numbers. Let us assume that $k \geqslant 1$ and $k \leqslant n$. Let x, y be finite sequences. The functor (k, n)-BitSubtracterOutput(x, y) yielding an element of InnerVertices(n-BitSubtracterStr(x, y)) is defined by:

(Def. 4) There exists a natural number i such that $k = i + 1$ and (k, n)-BitSubtracterOutput(x, y) = BitSubtracterOutput$(x(k), y(k),$ i-BitBorrowOutput$(x, y))$.

One can prove the following propositions:

(10) For all natural numbers n, k such that $k < n$ and for all finite sequences x, y holds $(k + 1, n)$-BitSubtracterOutput(x, y) = BitSubtracterOutput$(x(k + 1), y(k + 1), k$-BitBorrowOutput$(x, y))$.

(11) For every natural number n and for all finite sequences x, y holds InnerVertices(n-BitSubtracterStr(x, y)) is a binary relation.

(12) For all sets x, y, c holds InnerVertices(BorrowIStr(x, y, c)) = $\{\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, c\rangle, \mathrm{and}_2\rangle, \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle\}$.

(13) For all sets x, y, c such that $x \neq \langle\langle y, c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ holds InputVertices(BorrowIStr(x, y, c)) = $\{x, y, c\}$.

(14) For all sets x, y, c holds InnerVertices(BorrowStr(x, y, c)) = $\{\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, c\rangle, \mathrm{and}_2\rangle, \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle\} \cup \{$BorrowOutput$(x, y, c)\}$.

(15) For all sets x, y, c such that $x \neq \langle\langle y, c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ holds InputVertices(BorrowStr(x, y, c)) = $\{x, y, c\}$.

(16) For all sets x, y, c such that $x \neq \langle\langle y, c\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \mathrm{xor}\rangle$ holds InputVertices(BitSubtracterWithBorrowStr(x, y, c)) = $\{x, y, c\}$.

(17) For all sets x, y, c holds InnerVertices(BitSubtracterWithBorrowStr(x, y, c)) = $\{\langle\langle x, y\rangle, \mathrm{xor}\rangle$, 2GatesCircOutput$(x, y, c, \mathrm{xor})\} \cup \{\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, c\rangle, \mathrm{and}_2\rangle, \langle\langle x, c\rangle, \mathrm{and}_{2a}\rangle\} \cup \{$BorrowOutput$(x, y, c)\}$.

Let n be a natural number and let x, y be finite sequences. Observe that n-BitBorrowOutput(x, y) is pair.

The following propositions are true:

(18) Let x, y be finite sequences and n be a natural number. Then $(n$-BitBorrowOutput$(x, y))_{\mathbf{1}} = \varepsilon$ and $(n$-BitBorrowOutput$(x, y))_{\mathbf{2}} = \underline{Boolean^0 \longmapsto true}$ and $\pi_1((n$-BitBorrowOutput$(x, y))_{\mathbf{2}}) = Boolean^0$ or $\overline{(n\text{-BitBorrowOutput}(x, y))_{\mathbf{1}}} = 3$ and $(n$-BitBorrowOutput$(x, y))_{\mathbf{2}} = \mathrm{or}_3$ and $\pi_1((n$-BitBorrowOutput$(x, y))_{\mathbf{2}}) = Boolean^3$.

(19) Let n be a natural number, x, y be finite sequences, and p be a set. Then n-BitBorrowOutput$(x, y) \neq \langle p, \mathrm{and}_2\rangle$ and n-BitBorrowOutput$(x, y) \neq \langle p, \mathrm{and}_{2a}\rangle$ and n-BitBorrowOutput$(x, y) \neq \langle p, \mathrm{xor}\rangle$.

(20) Let f, g be nonpair yielding finite sequences and n be a natural number. Then InputVertices($(n + 1)$-BitSubtracterStr(f, g)) = InputVertices(n-BitSubtracterStr(f, g)) $\cup$ (InputVertices

(BitSubtracterWithBorrowStr($f(n+1)$, $g(n+1)$, n-BitBorrowOutput(f, g)))$\setminus$ $\{n$-BitBorrowOutput(f, g)$\}$) and InnerVertices(n-BitSubtracterStr(f, g)) is a binary relation and InputVertices(n-BitSubtracterStr(f, g)) has no pairs.

(21) For every natural number n and for all nonpair yielding finite sequences x, y with length n holds InputVertices(n-BitSubtracterStr(x, y)) $=$ rng $x \cup$ rng y.

(22) Let x, y, c be sets, s be a state of BorrowCirc(x, y, c), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x, y\rangle, \text{and}_{2a}\rangle)$ and $a_2 = s(\langle\langle y, c\rangle, \text{and}_2\rangle)$ and $a_3 = s(\langle\langle x, c\rangle, \text{and}_{2a}\rangle)$, then (Following($s$))(BorrowOutput($x, y, c$)) $= a_1 \vee a_2 \vee a_3$.

(23) Let x, y, c be sets. Suppose $x \neq \langle\langle y, c\rangle, \text{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \text{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \text{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \text{xor}\rangle$. Let s be a state of BorrowCirc(x, y, c). Then Following($s, 2$) is stable.

(24) Let x, y, c be sets. Suppose $x \neq \langle\langle y, c\rangle, \text{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \text{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \text{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \text{xor}\rangle$. Let s be a state of BitSubtracterWithBorrowCirc(x, y, c) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(c)$. Then (Following($s, 2$))(BitSubtracterOutput(x, y, c)) $= a_1 \oplus a_2 \oplus a_3$ and (Following($s, 2$))(BorrowOutput(x, y, c)) $= \neg a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee \neg a_1 \wedge a_3$.

(25) Let x, y, c be sets. Suppose $x \neq \langle\langle y, c\rangle, \text{and}_2\rangle$ and $y \neq \langle\langle x, c\rangle, \text{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \text{and}_{2a}\rangle$ and $c \neq \langle\langle x, y\rangle, \text{xor}\rangle$. Let s be a state of BitSubtracterWithBorrowCirc(x, y, c). Then Following($s, 2$) is stable.

(26) Let n be a natural number, x, y be nonpair yielding finite sequences with length n, and s be a state of n-BitSubtracterCirc(x, y). Then Following($s, 1 + 2 \cdot n$) is stable.

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