FORMALIZED MATHEMATICS
Volume 11, Number 3, 2003
University of Białystok

On the Kuratowski Closure-Complement Problem

Lilla Krystyna Bagińska
University of Białystok

Adam Grabowski[1]
University of Białystok

Summary. In this article we formalize the Kuratowski closure-complement result: there is at most 14 distinct sets that one can produce from a given subset A of a topological space T by applying closure and complement operators and that all 14 can be obtained from a suitable subset of $\mathbb{R}$, namely KuratExSet $= \{1\} \cup \mathbb{Q}(2,3) \cup (3,4) \cup (4,\infty)$.

The second part of the article deals with the maximal number of distinct sets which may be obtained from a given subset A of T by applying closure and interior operators. The subset KuratExSet of $\mathbb{R}$ is also enough to show that 7 can be achieved.

MML Identifier: `KURATO_1`.

The papers [15], [16], [10], [13], [11], [17], [14], [1], [3], [12], [7], [6], [8], [2], [4], [9], and [5] provide the notation and terminology for this paper.

1. Fourteen Kuratowski Sets

In this paper T is a non empty topological space and A is a subset of T.

The following proposition is true

(1) $\overline{-\overline{-\overline{-\overline{A}}}} = \overline{-\overline{A}}$.

Let us consider T, A. The functor Kurat14Part(A) is defined as follows:

(Def. 1) Kurat14Part(A) $= \{A, \overline{A}, -\overline{A}, \overline{-\overline{A}}, -\overline{-\overline{A}}, \overline{-\overline{-\overline{A}}}, -\overline{-\overline{-\overline{A}}}\}$.

Let us consider T, A. One can check that Kurat14Part(A) is finite.

Let us consider T, A. The functor Kurat14Set(A) yields a family of subsets of T and is defined by:

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102.

ISSN 1426–2630

(Def. 2) $\mathrm{Kurat14Set}(A) = \{A, \overline{A}, -\overline{A}, \overline{-\overline{A}}, -\overline{-\overline{A}}, \overline{-\overline{-\overline{A}}}, -\overline{-\overline{-\overline{A}}}\} \cup$
$\{-A, \overline{-A}, -\overline{-A}, \overline{-\overline{-A}}, -\overline{-\overline{-A}}, \overline{-\overline{-\overline{-A}}}, -\overline{-\overline{-\overline{-A}}}\}$.

We now state three propositions:

(2) $\mathrm{Kurat14Set}(A) = \mathrm{Kurat14Part}(A) \cup \mathrm{Kurat14Part}(-A)$.

(3) $A \in \mathrm{Kurat14Set}(A)$ and $\overline{A} \in \mathrm{Kurat14Set}(A)$ and $-\overline{A} \in \mathrm{Kurat14Set}(A)$ and $\overline{-\overline{A}} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-\overline{A}} \in \mathrm{Kurat14Set}(A)$ and $\overline{-\overline{-\overline{A}}} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-\overline{-\overline{A}}} \in \mathrm{Kurat14Set}(A)$.

(4) $-A \in \mathrm{Kurat14Set}(A)$ and $\overline{-A} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-A} \in \mathrm{Kurat14Set}(A)$ and $\overline{-\overline{-A}} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-\overline{-A}} \in \mathrm{Kurat14Set}(A)$ and $\overline{-\overline{-\overline{-A}}} \in \mathrm{Kurat14Set}(A)$ and $-\overline{-\overline{-\overline{-A}}} \in \mathrm{Kurat14Set}(A)$.

Let us consider T, A. The functor $\mathrm{Kurat14ClosedPart}(A)$ yielding a family of subsets of T is defined by:

(Def. 3) $\mathrm{Kurat14ClosedPart}(A) = \{\overline{A}, \overline{-\overline{A}}, \overline{-\overline{-\overline{A}}}, \overline{-A}, \overline{-\overline{-A}}, \overline{-\overline{-\overline{-A}}}\}$.

The functor $\mathrm{Kurat14OpenPart}(A)$ yields a family of subsets of T and is defined as follows:

(Def. 4) $\mathrm{Kurat14OpenPart}(A) = \{-\overline{A}, -\overline{-\overline{A}}, -\overline{-\overline{-\overline{A}}}, -\overline{-A}, -\overline{-\overline{-A}}, -\overline{-\overline{-\overline{-A}}}\}$.

We now state the proposition

(5) $\mathrm{Kurat14Set}(A) = \{A, -A\} \cup \mathrm{Kurat14ClosedPart}(A) \cup \mathrm{Kurat14OpenPart}(A)$.

Let us consider T, A. One can verify that $\mathrm{Kurat14Set}(A)$ is finite.

Next we state two propositions:

(6) For every subset Q of the carrier of T such that $Q \in \mathrm{Kurat14Set}(A)$ holds $-Q \in \mathrm{Kurat14Set}(A)$ and $\overline{Q} \in \mathrm{Kurat14Set}(A)$.

(7) $\mathrm{card}\,\mathrm{Kurat14Set}(A) \leqslant 14$.

2. Seven Kuratowski Sets

Let us consider T, A. The functor $\mathrm{Kurat7Set}(A)$ yielding a family of subsets of T is defined as follows:

(Def. 5) $\mathrm{Kurat7Set}(A) = \{A, \mathrm{Int}\,A, \overline{A}, \mathrm{Int}\,\overline{A}, \overline{\mathrm{Int}\,A}, \overline{\mathrm{Int}\,\overline{A}}, \mathrm{Int}\,\overline{\mathrm{Int}\,A}\}$.

We now state two propositions:

(8) $A \in \mathrm{Kurat7Set}(A)$ and $\mathrm{Int}\,A \in \mathrm{Kurat7Set}(A)$ and $\overline{A} \in \mathrm{Kurat7Set}(A)$ and $\mathrm{Int}\,\overline{A} \in \mathrm{Kurat7Set}(A)$ and $\overline{\mathrm{Int}\,A} \in \mathrm{Kurat7Set}(A)$ and $\overline{\mathrm{Int}\,\overline{A}} \in \mathrm{Kurat7Set}(A)$ and $\mathrm{Int}\,\overline{\mathrm{Int}\,A} \in \mathrm{Kurat7Set}(A)$.

(9) $\mathrm{Kurat7Set}(A) = \{A\} \cup \{\mathrm{Int}\,A, \mathrm{Int}\,\overline{A}, \mathrm{Int}\,\overline{\mathrm{Int}\,A}\} \cup \{\overline{A}, \overline{\mathrm{Int}\,A}, \overline{\mathrm{Int}\,\overline{A}}\}$.

Let us consider T, A. Note that $\mathrm{Kurat7Set}(A)$ is finite.

We now state two propositions:

(10) For every subset Q of the carrier of T such that $Q \in \mathrm{Kurat7Set}(A)$ holds $\mathrm{Int}\, Q \in \mathrm{Kurat7Set}(A)$ and $\overline{Q} \in \mathrm{Kurat7Set}(A)$.

(11) $\mathrm{card}\, \mathrm{Kurat7Set}(A) \leqslant 7$.

3. The Set Generating Exactly Fourteen Kuratowski Sets

The subset KuratExSet of $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 6) $\mathrm{KuratExSet} = \{1\} \cup]2,3[_{\mathbb{Q}} \cup]3,4[\cup]4,+\infty[$.

Next we state a number of propositions:

(12) $\overline{\mathrm{KuratExSet}} = \{1\} \cup [2,+\infty[$.

(13) $-\overline{\mathrm{KuratExSet}} =]-\infty,1[\cup]1,2[$.

(14) $\overline{-\overline{\mathrm{KuratExSet}}} =]-\infty,2]$.

(15) $-\overline{-\overline{\mathrm{KuratExSet}}} =]2,+\infty[$.

(16) $\overline{-\overline{-\overline{\mathrm{KuratExSet}}}} = [2,+\infty[$.

(17) $-\overline{-\overline{-\overline{\mathrm{KuratExSet}}}} =]-\infty,2[$.

(18) $-\mathrm{KuratExSet} =]-\infty,1[\cup]1,2] \cup]2,3[_{\mathbb{IQ}} \cup \{3\} \cup \{4\}$.

(19) $\overline{-\mathrm{KuratExSet}} =]-\infty,3] \cup \{4\}$.

(20) $-\overline{-\mathrm{KuratExSet}} =]3,4[\cup]4,+\infty[$.

(21) $\overline{-\overline{-\mathrm{KuratExSet}}} = [3,+\infty[$.

(22) $-\overline{-\overline{-\mathrm{KuratExSet}}} =]-\infty,3[$.

(23) $\overline{-\overline{-\overline{-\mathrm{KuratExSet}}}} =]-\infty,3]$.

(24) $-\overline{-\overline{-\overline{-\mathrm{KuratExSet}}}} =]3,+\infty[$.

4. The Set Generating Exactly Seven Kuratowski Sets

Next we state several propositions:

(25) $\mathrm{Int}\, \mathrm{KuratExSet} =]3,4[\cup]4,+\infty[$.

(26) $\overline{\mathrm{Int}\, \mathrm{KuratExSet}} = [3,+\infty[$.

(27) $\mathrm{Int}\, \overline{\mathrm{Int}\, \mathrm{KuratExSet}} =]3,+\infty[$.

(28) $\mathrm{Int}\, \overline{\mathrm{KuratExSet}} =]2,+\infty[$.

(29) $\overline{\mathrm{Int}\, \overline{\mathrm{KuratExSet}}} = [2,+\infty[$.

5. The Difference Between Chosen Kuratowski Sets

One can prove the following propositions:

(30) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \mathrm{Int}\,\overline{\mathrm{KuratExSet}}$.

(31) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \overline{\mathrm{KuratExSet}}$.

(32) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(33) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(34) $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(35) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \overline{\mathrm{KuratExSet}}$.

(36) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(37) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(38) $\mathrm{Int}\,\overline{\mathrm{KuratExSet}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(39) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \overline{\mathrm{KuratExSet}}$.

(40) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \overline{\mathrm{KuratExSet}}$.

(41) $\mathrm{Int}\,\mathrm{KuratExSet} \neq \overline{\mathrm{KuratExSet}}$.

(42) $\overline{\mathrm{KuratExSet}} \neq \mathrm{KuratExSet}$.

(43) $\mathrm{KuratExSet} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(44) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$.

(45) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

(46) $\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\mathrm{KuratExSet}$.

6. Final Proofs For Seven Sets

The following propositions are true:

(47) $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}} \neq \mathrm{Int}\,\overline{\mathrm{KuratExSet}}$.

(48) $\mathrm{Int}\,\mathrm{KuratExSet}$, $\mathrm{Int}\,\overline{\mathrm{KuratExSet}}$, $\mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$ are mutually different.

(49) $\overline{\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\mathrm{KuratExSet}}$, $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}$ are mutually different.

(50) For every set X such that $X \in \{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}\}$ holds X is an open non empty subset of $\mathbb{R}^{\mathbf{1}}$.

(51) For every set X such that $X \in \{\overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$ holds X is a closed subset of $\mathbb{R}^{\mathbf{1}}$.

(52) For every set X such that $X \in \{\mathrm{Int}\,\mathrm{KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int}\,\mathrm{KuratExSet}}\}$ holds $X \neq \mathbb{R}$.

(53) For every set X such that $X \in \{\overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\mathrm{KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$ holds $X \neq \mathbb{R}$.

(54) $\{\mathrm{Int\,KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int\,KuratExSet}}\}$ misses $\{\overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int\,KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$.

(55) $\mathrm{Int\,KuratExSet}$, $\mathrm{Int}\,\overline{\mathrm{KuratExSet}}$, $\mathrm{Int}\,\overline{\mathrm{Int\,KuratExSet}}$, $\overline{\mathrm{KuratExSet}}$, $\overline{\mathrm{Int\,KuratExSet}}$, $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}$ are mutually different.

Let us note that KuratExSet is non closed and non open.

Next we state three propositions:

(56) $\{\mathrm{Int\,KuratExSet}, \mathrm{Int}\,\overline{\mathrm{KuratExSet}}, \mathrm{Int}\,\overline{\mathrm{Int\,KuratExSet}}, \overline{\mathrm{KuratExSet}}, \overline{\mathrm{Int\,KuratExSet}}, \overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}\}$ misses $\{\mathrm{KuratExSet}\}$.

(57) KuratExSet, $\mathrm{Int\,KuratExSet}$, $\mathrm{Int}\,\overline{\mathrm{KuratExSet}}$, $\mathrm{Int}\,\overline{\mathrm{Int\,KuratExSet}}$, $\overline{\mathrm{KuratExSet}}$, $\overline{\mathrm{Int\,KuratExSet}}$, $\overline{\mathrm{Int}\,\overline{\mathrm{KuratExSet}}}$ are mutually different.

(58) card Kurat7Set(KuratExSet) = 7.

7. Final Proofs For Fourteen Sets

One can check that Kurat14ClosedPart(KuratExSet) has proper subsets and Kurat14OpenPart(KuratExSet) has proper subsets.

One can verify that Kurat14Set(KuratExSet) has proper subsets.

Let us note that Kurat14Set(KuratExSet) has non empty elements.

We now state the proposition

(59) For every set A with non empty elements and for every set B such that $B \subseteq A$ holds B has non empty elements.

Let us note that Kurat14ClosedPart(KuratExSet) has non empty elements and Kurat14OpenPart(KuratExSet) has non empty elements.

Let us note that there exists a family of subsets of $\mathbb{R}^{\mathbf{1}}$ which has proper subsets and non empty elements.

We now state the proposition

(60) Let F, G be families of subsets of $\mathbb{R}^{\mathbf{1}}$ with proper subsets and non empty elements. If F is open and G is closed, then F misses G.

Let us mention that Kurat14ClosedPart(KuratExSet) is closed and Kurat14OpenPart(KuratExSet) is open.

One can prove the following proposition

(61) Kurat14ClosedPart(KuratExSet) misses Kurat14OpenPart(KuratExSet).

Let us consider T, A. Observe that Kurat14ClosedPart(A) is finite and Kurat14OpenPart(A) is finite.

We now state three propositions:

(62) card Kurat14ClosedPart(KuratExSet) = 6.

(63) card Kurat14OpenPart(KuratExSet) = 6.

(64) {KuratExSet, −KuratExSet} misses Kurat14ClosedPart(KuratExSet).

Let us observe that KuratExSet is non empty.

The following three propositions are true:

(65) $\text{KuratExSet} \neq -\text{KuratExSet}$.

(66) $\{\text{KuratExSet}, -\text{KuratExSet}\}$ misses Kurat14OpenPart(KuratExSet).

(67) $\operatorname{card} \text{Kurat14Set}(\text{KuratExSet}) = 14$.

8. Properties of Kuratowski Sets

Let T be a topological structure and let A be a family of subsets of T. We say that A is closed for closure operator if and only if:

(Def. 7) For every subset P of the carrier of T such that $P \in A$ holds $\overline{P} \in A$.

We say that A is closed for interior operator if and only if:

(Def. 8) For every subset P of the carrier of T such that $P \in A$ holds $\operatorname{Int} P \in A$.

Let T be a 1-sorted structure and let A be a family of subsets of T. We say that A is closed for complement operator if and only if:

(Def. 9) For every subset P of the carrier of T such that $P \in A$ holds $-P \in A$.

Let us consider T, A. One can verify the following observations:

* Kurat14Set(A) is non empty,
* Kurat14Set(A) is closed for closure operator, and
* Kurat14Set(A) is closed for complement operator.

Let us consider T, A. One can check the following observations:

* Kurat7Set(A) is non empty,
* Kurat7Set(A) is closed for interior operator, and
* Kurat7Set(A) is closed for closure operator.

Let us consider T. One can check that there exists a family of subsets of T which is closed for interior operator, closed for closure operator, and non empty and there exists a family of subsets of T which is closed for complement operator, closed for closure operator, and non empty.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[3] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[4] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces - fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[5] Adam Grabowski. On the subcontinua of a real line. *Formalized Mathematics*, 11(**3**):313–322, 2003.
[6] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[7] Wojciech Leończuk and Krzysztof Prażmowski. Incidence projective spaces. *Formalized Mathematics*, 2(**2**):225–232, 1991.

[8] Yatsuka Nakamura. Half open intervals in real numbers. *Formalized Mathematics*, 10(**1**):21–22, 2002.
[9] Yatsuka Nakamura and Andrzej Trybulec. A mathematical model of CPU. *Formalized Mathematics*, 3(**2**):151–160, 1992.
[10] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[12] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[13] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics.*
[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[17] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received June 12, 2003