FORMALIZED MATHEMATICS
Volume 11, Number 3, 2003
University of Białystok

Real Linear Space of Real Sequences

Noboru Endou
Gifu National College of Technology

Yasumasa Suzuki
Take, Yokosuka-shi
Japan

Yasunari Shidama
Shinshu University
Nagano

Summary. The article is a continuation of [14]. As the example of real linear spaces, we introduce the arithmetic addition in the set of real sequences and also introduce the multiplication. This set has the arithmetic structure which depends on these arithmetic operations.

MML Identifier: RSSPACE.

The notation and terminology used here are introduced in the following papers: [12], [15], [5], [11], [6], [16], [2], [4], [3], [14], [13], [9], [8], [7], [10], and [1].

The non empty set the set of real sequences is defined by:

(Def. 1) For every set x holds $x \in$ the set of real sequences iff x is a sequence of real numbers.

Let a be a set. Let us assume that $a \in$ the set of real sequences. The functor $\mathrm{id}_{\mathrm{seq}}(a)$ yields a sequence of real numbers and is defined by:

(Def. 2) $\mathrm{id}_{\mathrm{seq}}(a) = a$.

Let a be a set. Let us assume that $a \in \mathbb{R}$. The functor $\mathrm{id}_{\mathbb{R}}(a)$ yielding a real number is defined by:

(Def. 3) $\mathrm{id}_{\mathbb{R}}(a) = a$.

We now state two propositions:

(1) There exists a binary operation A_1 on the set of real sequences such that for all elements a, b of the set of real sequences holds $A_1(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$ and A_1 is commutative and associative.

ISSN 1426–2630

(2) There exists a function f from $[: \mathbb{R}$, the set of real sequences $:]$ into the set of real sequences such that for all sets r, x if $r \in \mathbb{R}$ and $x \in$ the set of real sequences, then $f(\langle r, x \rangle) = \mathrm{id}_{\mathbb{R}}(r)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The binary operation $\mathrm{add}_{\mathrm{seq}}$ on the set of real sequences is defined as follows:

(Def. 4) For all elements a, b of the set of real sequences holds $\mathrm{add}_{\mathrm{seq}}(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$.

The function $\mathrm{mult}_{\mathrm{seq}}$ from $[: \mathbb{R}$, the set of real sequences $:]$ into the set of real sequences is defined by:

(Def. 5) For all sets r, x such that $r \in \mathbb{R}$ and $x \in$ the set of real sequences holds $\mathrm{mult}_{\mathrm{seq}}(\langle r, x \rangle) = \mathrm{id}_{\mathbb{R}}(r)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The element Zeroseq of the set of real sequences is defined by:

(Def. 6) For every natural number n holds $(\mathrm{id}_{\mathrm{seq}}(\mathrm{Zeroseq}))(n) = 0$.

One can prove the following propositions:

(3) For every sequence x of real numbers holds $\mathrm{id}_{\mathrm{seq}}(x) = x$.

(4) For all vectors v, w of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $v + w = \mathrm{id}_{\mathrm{seq}}(v) + \mathrm{id}_{\mathrm{seq}}(w)$.

(5) For every real number r and for every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $r \cdot v = r\ \mathrm{id}_{\mathrm{seq}}(v)$.

One can verify that $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ is Abelian.

We now state several propositions:

(6) For all vectors u, v, w of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(u + v) + w = u + (v + w)$.

(7) For every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $v + 0_{\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle} = v$.

(8) Let v be a vector of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$. Then there exists a vector w of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ such that $v + w = 0_{\langle \text{the set of real sequences}, \mathrm{Zeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle}$.

(9) For every real number a and for all vectors v, w of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $a \cdot (v + w) = a \cdot v + a \cdot w$.

(10) For all real numbers a, b and for every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(a + b) \cdot v = a \cdot v + b \cdot v$.

(11) For all real numbers a, b and for every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(a \cdot b) \cdot v = a \cdot (b \cdot v)$.

(12) For every vector v of $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$ holds $1 \cdot v = v$.

The real linear space the linear space of real sequences is defined by:

(Def. 7) The linear space of real sequences $=$ $\langle$the set of real sequences, Zeroseq, $\mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle$.

Let X be a real linear space and let X_1 be a subset of the carrier of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Add\_}(X_1, X)$ yielding a binary operation on X_1 is defined by:

(Def. 8) $\mathrm{Add\_}(X_1, X) = (\text{the addition of } X) \restriction [\![X_1, X_1]\!]$.

Let X be a real linear space and let X_1 be a subset of the carrier of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Mult\_}(X_1, X)$ yielding a function from $[\![\mathbb{R}, X_1]\!]$ into X_1 is defined as follows:

(Def. 9) $\mathrm{Mult\_}(X_1, X) = (\text{the external multiplication of } X) \restriction [\![\mathbb{R}, X_1]\!]$.

Let X be a real linear space and let X_1 be a subset of the carrier of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Zero\_}(X_1, X)$ yields an element of X_1 and is defined by:

(Def. 10) $\mathrm{Zero\_}(X_1, X) = 0_X$.

We now state the proposition

(13) Let V be a real linear space and V_1 be a subset of the carrier of V. Suppose V_1 is linearly closed and non empty. Then $\langle V_1, \mathrm{Zero\_}(V_1, V), \mathrm{Add\_}(V_1, V), \mathrm{Mult\_}(V_1, V) \rangle$ is a subspace of V.

The subset the set of l2-real sequences of the carrier of the linear space of real sequences is defined by the conditions (Def. 11).

(Def. 11)(i) The set of l2-real sequences is non empty, and

(ii) for every set x holds $x \in$ the set of l2-real sequences iff $x \in$ the set of real sequences and $\mathrm{id_{seq}}(x)\ \mathrm{id_{seq}}(x)$ is summable.

Next we state several propositions:

(14) The set of l2-real sequences is linearly closed and the set of l2-real sequences is non empty.

(15) $\langle$the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences)$\rangle$ is a subspace of the linear space of real sequences.

(16) $\langle$the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences)$\rangle$ is a real linear space.

(17)(i) The carrier of the linear space of real sequences = the set of real sequences,

(ii) for every set x holds x is an element of the carrier of the linear space of real sequences iff x is a sequence of real numbers,

(iii) for every set x holds x is a vector of the linear space of real sequences iff x is a sequence of real numbers,

(iv) for every vector u of the linear space of real sequences holds $u = \mathrm{id_{seq}}(u)$,

(v) for all vectors u, v of the linear space of real sequences holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$, and

(vi) for every real number r and for every vector u of the linear space of real sequences holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$.

(18) There exists a function f from [: the set of l2-real sequences, the set of l2-real sequences :] into $\mathbb{R}$ such that for all sets x, y if $x \in$ the set of l2-real sequences and $y \in$ the set of l2-real sequences, then $f(\langle x, y \rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(y))$.

The function $\mathrm{scalar}_{\mathrm{seq}}$ from [: the set of l2-real sequences, the set of l2-real sequences :] into $\mathbb{R}$ is defined by the condition (Def. 12).

(Def. 12) Let x, y be sets. Suppose $x \in$ the set of l2-real sequences and $y \in$ the set of l2-real sequences. Then $\mathrm{scalar}_{\mathrm{seq}}(\langle x, y \rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(y))$.

One can check that ⟨the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences), $\mathrm{scalar}_{\mathrm{seq}}$⟩ is non empty.

The non empty unitary space structure l2-Space is defined by the condition (Def. 13).

(Def. 13) l2-Space = ⟨the set of l2-real sequences, Zero_(the set of l2-real sequences, the linear space of real sequences), Add_(the set of l2-real sequences, the linear space of real sequences), Mult_(the set of l2-real sequences, the linear space of real sequences), $\mathrm{scalar}_{\mathrm{seq}}$⟩.

One can prove the following propositions:

(19) Let l be a unitary space structure. Suppose ⟨the carrier of l, the zero of l, the addition of l, the external multiplication of l⟩ is a real linear space. Then l is a real linear space.

(20) Let r_1 be a sequence of real numbers. If for every natural number n holds $r_1(n) = 0$, then r_1 is summable and $\sum r_1 = 0$.

(21) Let r_1 be a sequence of real numbers. Suppose for every natural number n holds $0 \leqslant r_1(n)$ and r_1 is summable and $\sum r_1 = 0$. Let n be a natural number. Then $r_1(n) = 0$.

Let us observe that l2-Space is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[7] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[8] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[9] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(**4**):511–516, 1991.
[10] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(**4**):449–452, 1991.
[11] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics.*
[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[13] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.
[14] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[16] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received April 3, 2003