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Hilbert Space of Real Sequences

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Summary. A continuation of [16]. As the example of real unitary spaces, we introduce the arithmetic addition and multiplication in the set of square sum able real sequences and introduce the scaler products also. This set has the structure of the Hilbert space.

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The articles [15], [17], [3], [14], [5], [18], [1], [2], [16], [12], [9], [10], [11], [8], [6], [7], [13], and [4] provide the terminology and notation for this paper.

1. Hilbert Space of Real Sequences

One can prove the following two propositions:

(1) The carrier of l2-Space = the set of l2-real sequences and for every set x holds x is an element of the carrier of l2-Space iff x is a sequence of real numbers and $\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(x)$ is summable and for every set x holds x is a vector of l2-Space iff x is a sequence of real numbers and $\mathrm{id}_{\mathrm{seq}}(x)\ \mathrm{id}_{\mathrm{seq}}(x)$ is summable and $0_{\text{l2-Space}} = \mathrm{Zeroseq}$ and for every vector u of l2-Space holds $u = \mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of l2-Space holds $u+v = \mathrm{id}_{\mathrm{seq}}(u)+\mathrm{id}_{\mathrm{seq}}(v)$ and for every real number r and for every vector u of l2-Space holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$ and for every vector u of l2-Space holds $-u = -\mathrm{id}_{\mathrm{seq}}(u)$ and $\mathrm{id}_{\mathrm{seq}}(-u) = -\mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of l2-Space holds $u - v = \mathrm{id}_{\mathrm{seq}}(u) - \mathrm{id}_{\mathrm{seq}}(v)$ and for all vectors v, w of

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l2-Space holds $\mathrm{id}_{\mathrm{seq}}(v)\ \mathrm{id}_{\mathrm{seq}}(w)$ is summable and for all vectors v, w of l2-Space holds $(v|w) = \sum(\mathrm{id}_{\mathrm{seq}}(v)\ \mathrm{id}_{\mathrm{seq}}(w))$.

(2) Let x, y, z be points of l2-Space and a be a real number. Then $(x|x) = 0$ iff $x = 0_{\text{l2-Space}}$ and $0 \leqslant (x|x)$ and $(x|y) = (y|x)$ and $((x + y)|z) = (x|z) + (y|z)$ and $((a \cdot x)|y) = a \cdot (x|y)$.

Let us note that l2-Space is real unitary space-like.

One can prove the following proposition

(3) For every sequence v_1 of l2-Space such that v_1 is a Cauchy sequence holds v_1 is convergent.

Let us mention that l2-Space is Hilbert and complete.

2. Miscellaneous

We now state several propositions:

(4) Let r_1 be a sequence of real numbers. Suppose for every natural number n holds $0 \leqslant r_1(n)$ and r_1 is summable. Then

(i) for every natural number n holds $r_1(n) \leqslant (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$,

(ii) for every natural number n holds $0 \leqslant (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$,

(iii) for every natural number n holds $(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \leqslant \sum r_1$, and

(iv) for every natural number n holds $r_1(n) \leqslant \sum r_1$.

(5) For all real numbers x, y holds $(x + y) \cdot (x + y) \leqslant 2 \cdot x \cdot x + 2 \cdot y \cdot y$ and for all real numbers x, y holds $x \cdot x \leqslant 2 \cdot (x - y) \cdot (x - y) + 2 \cdot y \cdot y$.

(6) Let e be a real number and s_1 be a sequence of real numbers. Suppose s_1 is convergent and there exists a natural number k such that for every natural number i such that $k \leqslant i$ holds $s_1(i) \leqslant e$. Then $\lim s_1 \leqslant e$.

(7) Let c be a real number and s_1 be a sequence of real numbers. Suppose s_1 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number i holds $r_1(i) = (s_1(i) - c) \cdot (s_1(i) - c)$. Then r_1 is convergent and $\lim r_1 = (\lim s_1 - c) \cdot (\lim s_1 - c)$.

(8) Let c be a real number and s_1, s_2 be sequences of real numbers. Suppose s_1 is convergent and s_2 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number i holds $r_1(i) = (s_1(i) - c) \cdot (s_1(i) - c) + s_2(i)$. Then r_1 is convergent and $\lim r_1 = (\lim s_1 - c) \cdot (\lim s_1 - c) + \lim s_2$.

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