

On Semilattice Structure of Mizar Types

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Summary. The aim of this paper is to develop a formal theory of Mizar types. The presented theory is an approach to the structure of Mizar types as a sup-semilattice with widening (subtyping) relation as the order. It is an abstraction from the existing implementation of the Mizar verifier and formalization of the ideas from [9].

MML Identifier: ABCMIZ_0.

The articles [20], [14], [24], [26], [23], [25], [3], [21], [1], [11], [12], [16], [10], [13], [18], [15], [4], [2], [19], [22], [5], [6], [7], [8], and [17] provide the terminology and notation for this paper.

1. SEMILATTICE OF WIDENING

Let us mention that every non empty relational structure which is trivial and reflexive is also complete.

Let T be a relational structure. A type of T is an element of T .

Let T be a relational structure. We say that T is Noetherian if and only if:

(Def. 1) The internal relation of T is reversely well founded.

Let us observe that every non empty relational structure which is trivial is also Noetherian.

Let T be a non empty relational structure. Let us observe that T is Noetherian if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let A be a non empty subset of T . Then there exists an element a of T such that $a \in A$ and for every element b of T such that $b \in A$ holds $a \not\prec b$.

Let T be a poset. We say that T is Mizar-widening-like if and only if:

(Def. 3) T is a sup-semilattice and Noetherian.

Let us mention that every poset which is Mizar-widening-like is also Noetherian and upper-bounded and has l.u.b.'s.

Let us note that every sup-semilattice which is Noetherian is also Mizar-widening-like.

Let us observe that there exists a complete sup-semilattice which is Mizar-widening-like.

Let T be a Noetherian relational structure. One can check that the internal relation of T is reversely well founded.

Next we state the proposition

- (1) For every Noetherian sup-semilattice T and for every ideal I of T holds $\sup I$ exists in T and $\sup I \in I$.

2. ADJECTIVES

We consider adjective structures as systems

\langle a set of adjectives, an operation non \rangle ,

where the set of adjectives is a set and the operation non is a unary operation on the set of adjectives.

Let A be an adjective structure. We say that A is void if and only if:

- (Def. 4) The set of adjectives of A is empty.

An adjective of A is an element of the set of adjectives of A .

The following proposition is true

- (2) Let A_1, A_2 be adjective structures. Suppose the set of adjectives of $A_1 =$ the set of adjectives of A_2 . If A_1 is void, then A_2 is void.

Let A be an adjective structure and let a be an element of the set of adjectives of A . The functor $\text{non } a$ yields an adjective of A and is defined as follows:

- (Def. 5) $\text{non } a = (\text{the operation non of } A)(a)$.

One can prove the following proposition

- (3) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2 . Let a_1 be an adjective of A_1 and a_2 be an adjective of A_2 . If $a_1 = a_2$, then $\text{non } a_1 = \text{non } a_2$.

Let A be an adjective structure. We say that A is involutive if and only if:

- (Def. 6) For every adjective a of A holds $\text{non non } a = a$.

We say that A is without fixpoints if and only if:

- (Def. 7) It is not true that there exists an adjective a of A such that $\text{non } a = a$.

We now state three propositions:

- (4) Let a_1, a_2 be sets. Suppose $a_1 \neq a_2$. Let A be an adjective structure. Suppose the set of adjectives of $A = \{a_1, a_2\}$ and $(\text{the operation non of } A)(a_1) = a_2$ and $(\text{the operation non of } A)(a_2) = a_1$. Then A is non void, involutive, and without fixpoints.

- (5) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2 . If A_1 is involutive, then A_2 is involutive.
- (6) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2 . If A_1 is without fixpoints, then A_2 is without fixpoints.

Let us observe that there exists a strict adjective structure which is non void, involutive, and without fixpoints.

Let A be a non void adjective structure. Observe that the set of adjectives of A is non empty.

We consider TA -structures as extensions of relational structure and adjective structure as systems

\langle a carrier, a set of adjectives, an internal relation, an operation non, an adjective map \rangle ,

where the carrier and the set of adjectives are sets, the internal relation is a binary relation on the carrier, the operation non is a unary operation on the set of adjectives, and the adjective map is a function from the carrier into Fin the set of adjectives.

Let X be a non empty set, let A be a set, let r be a binary relation on X , let n be a unary operation on A , and let a be a function from X into Fin A . Observe that $\langle X, A, r, n, a \rangle$ is non empty.

Let X be a set, let A be a non empty set, let r be a binary relation on X , let n be a unary operation on A , and let a be a function from X into Fin A . One can check that $\langle X, A, r, n, a \rangle$ is non void.

One can check that there exists a TA -structure which is trivial, reflexive, non empty, non void, involutive, without fixpoints, and strict.

Let T be a TA -structure and let t be an element of T . The functor $\text{adjs } t$ yields a subset of the set of adjectives of T and is defined as follows:

(Def. 8) $\text{adjs } t = (\text{the adjective map of } T)(t)$.

One can prove the following proposition

- (7) Let T_1, T_2 be TA -structures. Suppose the TA -structure of $T_1 =$ the TA -structure of T_2 . Let t_1 be a type of T_1 and t_2 be a type of T_2 . If $t_1 = t_2$, then $\text{adjs } t_1 = \text{adjs } t_2$.

Let T be a TA -structure. We say that T is consistent if and only if:

(Def. 9) For every type t of T and for every adjective a of T such that $a \in \text{adjs } t$ holds $\text{non } a \notin \text{adjs } t$.

Next we state the proposition

- (8) Let T_1, T_2 be TA -structures. Suppose the TA -structure of $T_1 =$ the TA -structure of T_2 . If T_1 is consistent, then T_2 is consistent.

Let T be a non empty TA -structure. We say that T has structured adjectives if and only if:

(Def. 10) The adjective map of T is a join-preserving map from T into $(2_{\subseteq}^{\text{the set of adjectives of } T})_{\text{op}}$.

We now state the proposition

(9) Let T_1, T_2 be non empty TA -structures. Suppose the TA -structure of $T_1 =$ the TA -structure of T_2 . If T_1 has structured adjectives, then T_2 has structured adjectives.

Let T be a reflexive transitive antisymmetric TA -structure with l.u.b.'s. Let us observe that T has structured adjectives if and only if:

(Def. 11) For all types t_1, t_2 of T holds $\text{adjs}(t_1 \sqcup t_2) = \text{adjs } t_1 \cap \text{adjs } t_2$.

One can prove the following proposition

(10) Let T be a reflexive transitive antisymmetric TA -structure with l.u.b.'s. Suppose T has structured adjectives. Let t_1, t_2 be types of T . If $t_1 \leq t_2$, then $\text{adjs } t_2 \subseteq \text{adjs } t_1$.

Let T be a TA -structure and let a be an element of the set of adjectives of T . The functor types a yields a subset of T and is defined as follows:

(Def. 12) For every set x holds $x \in \text{types } a$ iff there exists a type t of T such that $x = t$ and $a \in \text{adjs } t$.

Let T be a non empty TA -structure and let A be a subset of the set of adjectives of T . The functor types A yielding a subset of T is defined as follows:

(Def. 13) For every type t of T holds $t \in \text{types } A$ iff for every adjective a of T such that $a \in A$ holds $t \in \text{types } a$.

One can prove the following propositions:

(11) Let T_1, T_2 be TA -structures. Suppose the TA -structure of $T_1 =$ the TA -structure of T_2 . Let a_1 be an adjective of T_1 and a_2 be an adjective of T_2 . If $a_1 = a_2$, then $\text{types } a_1 = \text{types } a_2$.

(12) For every non empty TA -structure T and for every adjective a of T holds $\text{types } a = \{t; t \text{ ranges over types of } T: a \in \text{adjs } t\}$.

(13) Let T be a TA -structure, t be a type of T , and a be an adjective of T . Then $a \in \text{adjs } t$ if and only if $t \in \text{types } a$.

(14) Let T be a non empty TA -structure, t be a type of T , and A be a subset of the set of adjectives of T . Then $A \subseteq \text{adjs } t$ if and only if $t \in \text{types } A$.

(15) For every non void TA -structure T and for every type t of T holds $\text{adjs } t = \{a; a \text{ ranges over adjectives of } T: t \in \text{types } a\}$.

(16) Let T be a non empty TA -structure and t be a type of T . Then $\text{types}(\emptyset_{\text{the set of adjectives of } T}) = \text{the carrier of } T$.

Let T be a TA -structure. We say that T has typed adjectives if and only if:

(Def. 14) For every adjective a of T holds $\text{types } a \cup \text{types non } a$ is non empty.

We now state the proposition

- (17) Let T_1, T_2 be TA -structures. Suppose the TA -structure of $T_1 =$ the TA -structure of T_2 . If T_1 has typed adjectives, then T_2 has typed adjectives.

Let us mention that there exists a complete upper-bounded non empty trivial reflexive transitive antisymmetric strict TA -structure which is non void, Mizar-widening-like, involutive, without fixpoints, and consistent and has structured adjectives and typed adjectives.

Next we state the proposition

- (18) For every consistent TA -structure T and for every adjective a of T holds types a misses types non a .

Let T be a reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives and let a be an adjective of T . Note that types a is lower and directed.

Let T be a reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives and let A be a subset of the set of adjectives of T . One can verify that types A is lower and directed.

We now state the proposition

- (19) Let T be reflexive antisymmetric transitive TA -structure with l.u.b.'s with structured adjectives and a be an adjective of T . Then types a is empty or types a is an ideal of T .

3. APPLICABILITY OF ADJECTIVES

Let T be a TA -structure, let t be an element of T , and let a be an adjective of T . We say that a is applicable to t if and only if:

- (Def. 15) There exists a type t' of T such that $t' \in$ types a and $t' \leq t$.

Let T be a TA -structure, let t be a type of T , and let A be a subset of the set of adjectives of T . We say that A is applicable to t if and only if:

- (Def. 16) There exists a type t' of T such that $A \subseteq$ adjs t' and $t' \leq t$.

We now state the proposition

- (20) Let T be a reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, a be an adjective of T , and t be a type of T . If a is applicable to t , then types $a \cap \downarrow t$ is an ideal of T .

Let T be a non empty reflexive transitive TA -structure, let t be an element of T , and let a be an adjective of T . The functor $a * t$ yielding a type of T is defined by:

- (Def. 17) $a * t = \sup(\text{types } a \cap \downarrow t)$.

The following propositions are true:

- (21) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and a be an adjective of T . If a is applicable to t , then $a * t \leq t$.
- (22) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and a be an adjective of T . If a is applicable to t , then $a \in \text{adjs}(a * t)$.
- (23) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and a be an adjective of T . If a is applicable to t , then $a * t \in \text{types } a$.
- (24) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , a be an adjective of T , and t' be a type of T . If $t' \leq t$ and $a \in \text{adjs } t'$, then a is applicable to t and $t' \leq a * t$.
- (25) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and a be an adjective of T . If $a \in \text{adjs } t$, then a is applicable to t and $a * t = t$.
- (26) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and a, b be adjectives of T . Suppose a is applicable to t and b is applicable to $a * t$. Then b is applicable to t and a is applicable to $b * t$ and $a * (b * t) = b * (a * t)$.
- (27) Let T be a reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, A be a subset of the set of adjectives of T , and t be a type of T . If A is applicable to t , then $\text{types } A \cap \downarrow t$ is an ideal of T .

Let T be a non empty reflexive transitive TA -structure, let t be a type of T , and let A be a subset of the set of adjectives of T . The functor $A * t$ yielding a type of T is defined as follows:

(Def. 18) $A * t = \text{sup}(\text{types } A \cap \downarrow t)$.

Next we state the proposition

- (28) Let T be a non empty reflexive transitive antisymmetric TA -structure and t be a type of T . Then $\emptyset_{\text{the set of adjectives of } T} * t = t$.

Let T be a non empty non void reflexive transitive TA -structure, let t be a type of T , and let p be a finite sequence of elements of the set of adjectives of T . The functor $\text{apply}(p, t)$ yielding a finite sequence of elements of the carrier of T is defined by the conditions (Def. 19).

- (Def. 19)(i) $\text{len } \text{apply}(p, t) = \text{len } p + 1$,
- (ii) $(\text{apply}(p, t))(1) = t$, and
- (iii) for every natural number i and for every adjective a of T and for every type t of T such that $i \in \text{dom } p$ and $a = p(i)$ and $t = (\text{apply}(p, t))(i)$ holds $(\text{apply}(p, t))(i + 1) = a * t$.

Let T be a non empty non void reflexive transitive TA -structure, let t be a type of T , and let p be a finite sequence of elements of the set of adjectives of T . Note that $\text{apply}(p, t)$ is non empty.

One can prove the following two propositions:

- (29) Let T be a non empty non void reflexive transitive TA -structure and t be a type of T . Then $\text{apply}(\varepsilon_{(\text{the set of adjectives of } T)}, t) = \langle t \rangle$.
- (30) Let T be a non empty non void reflexive transitive TA -structure, t be a type of T , and a be an adjective of T . Then $\text{apply}(\langle a \rangle, t) = \langle t, a * t \rangle$.

Let T be a non empty non void reflexive transitive TA -structure, let t be a type of T , and let v be a finite sequence of elements of the set of adjectives of T . The functor $v * t$ yielding a type of T is defined by:

(Def. 20) $v * t = (\text{apply}(v, t))(\text{len } v + 1)$.

The following propositions are true:

- (31) Let T be a non empty non void reflexive transitive TA -structure and t be a type of T . Then $\varepsilon_{(\text{the set of adjectives of } T)} * t = t$.
- (32) Let T be a non empty non void reflexive transitive TA -structure, t be a type of T , and a be an adjective of T . Then $\langle a \rangle * t = a * t$.
- (33) For all finite sequences p, q and for every natural number i such that $i \geq 1$ and $i < \text{len } p$ holds $(p \text{ }^{\text{s}}\text{ }^{\wedge} q)(i) = p(i)$.
- (34) Let p be a non empty finite sequence, q be a finite sequence, and i be a natural number. If $i < \text{len } q$, then $(p \text{ }^{\text{s}}\text{ }^{\wedge} q)(\text{len } p + i) = q(i + 1)$.
- (35) Let T be a non empty non void reflexive transitive TA -structure, t be a type of T , and v_1, v_2 be finite sequences of elements of the set of adjectives of T . Then $\text{apply}(v_1 \wedge v_2, t) = (\text{apply}(v_1, t)) \text{ }^{\text{s}}\text{ }^{\wedge} \text{apply}(v_2, v_1 * t)$.
- (36) Let T be a non empty non void reflexive transitive TA -structure, t be a type of T , v_1, v_2 be finite sequences of elements of the set of adjectives of T , and i be a natural number. If $i \in \text{dom } v_1$, then $(\text{apply}(v_1 \wedge v_2, t))(i) = (\text{apply}(v_1, t))(i)$.
- (37) Let T be a non empty non void reflexive transitive TA -structure, t be a type of T , and v_1, v_2 be finite sequences of elements of the set of adjectives of T . Then $(\text{apply}(v_1 \wedge v_2, t))(\text{len } v_1 + 1) = v_1 * t$.
- (38) Let T be a non empty non void reflexive transitive TA -structure, t be a type of T , and v_1, v_2 be finite sequences of elements of the set of adjectives of T . Then $v_2 * (v_1 * t) = (v_1 \wedge v_2) * t$.

Let T be a non empty non void reflexive transitive TA -structure, let t be a type of T , and let v be a finite sequence of elements of the set of adjectives of T .

We say that v is applicable to t if and only if the condition (Def. 21) is satisfied.

- (Def. 21) Let i be a natural number, a be an adjective of T , and s be a type of T . If $i \in \text{dom } v$ and $a = v(i)$ and $s = (\text{apply}(v, t))(i)$, then a is applicable to s .

Next we state a number of propositions:

- (39) Let T be a non empty non void reflexive transitive TA -structure and t be a type of T . Then $\varepsilon_{(\text{the set of adjectives of } T)}$ is applicable to t .
- (40) Let T be a non empty non void reflexive transitive TA -structure, t be a type of T , and a be an adjective of T . Then a is applicable to t if and only if $\langle a \rangle$ is applicable to t .
- (41) Let T be a non empty non void reflexive transitive TA -structure, t be a type of T , and v_1, v_2 be finite sequences of elements of the set of adjectives of T . Suppose $v_1 \hat{\ } v_2$ is applicable to t . Then v_1 is applicable to t and v_2 is applicable to $v_1 * t$.
- (42) Let T be a Noetherian reflexive transitive antisymmetric non void TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and v be a finite sequence of elements of the set of adjectives of T . Suppose v is applicable to t . Let i_1, i_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1 \leq i_2$ and $i_2 \leq \text{len } v + 1$. Let t_1, t_2 be types of T . If $t_1 = (\text{apply}(v, t))(i_1)$ and $t_2 = (\text{apply}(v, t))(i_2)$, then $t_2 \leq t_1$.
- (43) Let T be a Noetherian reflexive transitive antisymmetric non void TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and v be a finite sequence of elements of the set of adjectives of T . Suppose v is applicable to t . Let s be a type of T . If $s \in \text{rng apply}(v, t)$, then $v * t \leq s$ and $s \leq t$.
- (44) Let T be a Noetherian reflexive transitive antisymmetric non void TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and v be a finite sequence of elements of the set of adjectives of T . If v is applicable to t , then $v * t \leq t$.
- (45) Let T be a Noetherian reflexive transitive antisymmetric non void TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and v be a finite sequence of elements of the set of adjectives of T . If v is applicable to t , then $\text{rng } v \subseteq \text{adjs}(v * t)$.
- (46) Let T be a Noetherian reflexive transitive antisymmetric non void TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and v be a finite sequence of elements of the set of adjectives of T . Suppose v is applicable to t . Let A be a subset of the set of adjectives of T . If $A = \text{rng } v$, then A is applicable to t .
- (47) Let T be a Noetherian reflexive transitive antisymmetric non void TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and v_1, v_2 be finite sequences of elements of the set of adjectives of T . Suppose v_1 is applicable to t and $\text{rng } v_2 \subseteq \text{rng } v_1$. Let s be a type of T . If $s \in \text{rng apply}(v_2, t)$, then $v_1 * t \leq s$.
- (48) Let T be a Noetherian reflexive transitive antisymmetric non void TA -

- structure with l.u.b.'s with structured adjectives, t be a type of T , and v_1, v_2 be finite sequences of elements of the set of adjectives of T . If $v_1 \wedge v_2$ is applicable to t , then $v_2 \wedge v_1$ is applicable to t .
- (49) Let T be a Noetherian reflexive transitive antisymmetric non void TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and v_1, v_2 be finite sequences of elements of the set of adjectives of T . If $v_1 \wedge v_2$ is applicable to t , then $(v_1 \wedge v_2) * t = (v_2 \wedge v_1) * t$.
- (50) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and A be a subset of the set of adjectives of T . If A is applicable to t , then $A * t \leq t$.
- (51) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and A be a subset of the set of adjectives of T . If A is applicable to t , then $A \subseteq \text{ads}(A * t)$.
- (52) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and A be a subset of the set of adjectives of T . If A is applicable to t , then $A * t \in \text{types } A$.
- (53) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , A be a subset of the set of adjectives of T , and t' be a type of T . If $t' \leq t$ and $A \subseteq \text{ads } t'$, then A is applicable to t and $t' \leq A * t$.
- (54) Let T be a Noetherian reflexive transitive antisymmetric TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and A be a subset of the set of adjectives of T . If $A \subseteq \text{ads } t$, then A is applicable to t and $A * t = t$.
- (55) Let T be a TA -structure, t be a type of T , and A, B be subsets of the set of adjectives of T . If A is applicable to t and $B \subseteq A$, then B is applicable to t .
- (56) Let T be a Noetherian reflexive transitive antisymmetric non void TA -structure with l.u.b.'s with structured adjectives, t be a type of T , a be an adjective of T , and A, B be subsets of the set of adjectives of T . If $B = A \cup \{a\}$ and B is applicable to t , then $a * (A * t) = B * t$.
- (57) Let T be a Noetherian reflexive transitive antisymmetric non void TA -structure with l.u.b.'s with structured adjectives, t be a type of T , and v be a finite sequence of elements of the set of adjectives of T . Suppose v is applicable to t . Let A be a subset of the set of adjectives of T . If $A = \text{rng } v$, then $v * t = A * t$.

4. SUBJECT FUNCTION

Let T be a non empty non void TA -structure. The functor $\text{sub } T$ yields a function from the set of adjectives of T into the carrier of T and is defined as follows:

(Def. 22) For every adjective a of T holds $(\text{sub } T)(a) = \text{sup}(\text{types } a \cup \text{types non } a)$.

We introduce TAS -structures which are extensions of TA -structure and are systems

\langle a carrier, a set of adjectives, an internal relation, an operation non , an adjective map, a subject map \rangle ,

where the carrier and the set of adjectives are sets, the internal relation is a binary relation on the carrier, the operation non is a unary operation on the set of adjectives, the adjective map is a function from the carrier into Fin the set of adjectives, and the subject map is a function from the set of adjectives into the carrier.

Let us observe that there exists a TAS -structure which is non void, reflexive, trivial, non empty, and strict.

Let T be a non empty non void TAS -structure and let a be an adjective of T . The functor $\text{sub } a$ yields a type of T and is defined as follows:

(Def. 23) $\text{sub } a = (\text{the subject map of } T)(a)$.

Let T be a non empty non void TAS -structure. We say that T is absorbing non if and only if:

(Def. 24) $(\text{The subject map of } T) \cdot (\text{the operation non of } T) = \text{the subject map of } T$.

We say that T is subjected if and only if:

(Def. 25) For every adjective a of T holds $\text{types } a \cup \text{types non } a \leq \text{sub } a$ and if $\text{types } a \neq \emptyset$ and $\text{types non } a \neq \emptyset$, then $\text{sub } a = \text{sup}(\text{types } a \cup \text{types non } a)$.

Let T be a non empty non void TAS -structure. Let us observe that T is absorbing non if and only if:

(Def. 26) For every adjective a of T holds $\text{sub non } a = \text{sub } a$.

Let T be a non empty non void TAS -structure, let t be an element of T , and let a be an adjective of T . We say that a is properly applicable to t if and only if:

(Def. 27) $t \leq \text{sub } a$ and a is applicable to t .

Let T be a non empty non void reflexive transitive TAS -structure, let t be a type of T , and let v be a finite sequence of elements of the set of adjectives of T .

We say that v is properly applicable to t if and only if the condition (Def. 28) is satisfied.

(Def. 28) Let i be a natural number, a be an adjective of T , and s be a type of T . If $i \in \text{dom } v$ and $a = v(i)$ and $s = (\text{apply}(v, t))(i)$, then a is properly

applicable to s .

One can prove the following propositions:

- (58) Let T be a non empty non void reflexive transitive TAS -structure, t be a type of T , and v be a finite sequence of elements of the set of adjectives of T . If v is properly applicable to t , then v is applicable to t .
- (59) Let T be a non empty non void reflexive transitive TAS -structure and t be a type of T . Then $\varepsilon_{(\text{the set of adjectives of } T)}$ is properly applicable to t .
- (60) Let T be a non empty non void reflexive transitive TAS -structure, t be a type of T , and a be an adjective of T . Then a is properly applicable to t if and only if $\langle a \rangle$ is properly applicable to t .
- (61) Let T be a non empty non void reflexive transitive TAS -structure, t be a type of T , and v_1, v_2 be finite sequences of elements of the set of adjectives of T . Suppose $v_1 \wedge v_2$ is properly applicable to t . Then v_1 is properly applicable to t and v_2 is properly applicable to $v_1 * t$.
- (62) Let T be a non empty non void reflexive transitive TAS -structure, t be a type of T , and v_1, v_2 be finite sequences of elements of the set of adjectives of T . Suppose v_1 is properly applicable to t and v_2 is properly applicable to $v_1 * t$. Then $v_1 \wedge v_2$ is properly applicable to t .

Let T be a non empty non void reflexive transitive TAS -structure, let t be a type of T , and let A be a subset of the set of adjectives of T . We say that A is properly applicable to t if and only if the condition (Def. 29) is satisfied.

- (Def. 29) There exists a finite sequence s of elements of the set of adjectives of T such that $\text{rng } s = A$ and s is properly applicable to t .

Next we state two propositions:

- (63) Let T be a non empty non void reflexive transitive TAS -structure, t be a type of T , and A be a subset of the set of adjectives of T . If A is properly applicable to t , then A is finite.
- (64) Let T be a non empty non void reflexive transitive TAS -structure and t be a type of T . Then $\emptyset_{\text{the set of adjectives of } T}$ is properly applicable to t .

The scheme *MinimalFiniteSet* concerns a unary predicate \mathcal{P} , and states that:

There exists a finite set A such that $\mathcal{P}[A]$ and for every set B such that $B \subseteq A$ and $\mathcal{P}[B]$ holds $B = A$

provided the following requirement is met:

- There exists a finite set A such that $\mathcal{P}[A]$.

One can prove the following proposition

- (65) Let T be a non empty non void reflexive transitive TAS -structure, t be a type of T , and A be a subset of the set of adjectives of T . Suppose A is properly applicable to t . Then there exists a subset B of the set of adjectives of T such that
- (i) $B \subseteq A$,

- (ii) B is properly applicable to t ,
- (iii) $A * t = B * t$, and
- (iv) for every subset C of the set of adjectives of T such that $C \subseteq B$ and C is properly applicable to t and $A * t = C * t$ holds $C = B$.

Let T be a non empty non void reflexive transitive TAS -structure. We say that T is commutative if and only if the condition (Def. 30) is satisfied.

- (Def. 30) Let t_1, t_2 be types of T and a be an adjective of T . Suppose a is properly applicable to t_1 and $a * t_1 \leq t_2$. Then there exists a finite subset A of the set of adjectives of T such that A is properly applicable to $t_1 \sqcup t_2$ and $A * (t_1 \sqcup t_2) = t_2$.

Let us observe that there exists a complete upper-bounded non empty non void trivial reflexive transitive antisymmetric strict TAS -structure which is Mizar-widening-like, involutive, without fixpoints, consistent, absorbing non, subjected, and commutative and has structured adjectives and typed adjectives.

Next we state the proposition

- (66) Let T be a Noetherian reflexive transitive antisymmetric non void TAS -structure with l.u.b.'s with structured adjectives, t be a type of T , and A be a subset of the set of adjectives of T . Suppose A is properly applicable to t . Then there exists an one-to-one finite sequence s of elements of the set of adjectives of T such that $\text{rng } s = A$ and s is properly applicable to t .

5. REDUCTION OF ADJECTIVES

Let T be a non empty non void reflexive transitive TAS -structure. The functor $\circ \rightarrow_T$ yields a binary relation on T and is defined by the condition (Def. 31).

- (Def. 31) Let t_1, t_2 be types of T . Then $\langle t_1, t_2 \rangle \in \circ \rightarrow_T$ if and only if there exists an adjective a of T such that $a \notin \text{ads } t_2$ and a is properly applicable to t_2 and $a * t_2 = t_1$.

Next we state the proposition

- (67) Let T be an antisymmetric non void reflexive transitive Noetherian TAS -structure with l.u.b.'s with structured adjectives. Then $\circ \rightarrow_T \subseteq$ the internal relation of T .

The scheme *RedInd* deals with a non empty set \mathcal{A} , a binary relation \mathcal{B} on \mathcal{A} , and a binary predicate \mathcal{P} , and states that:

For all elements x, y of \mathcal{A} such that \mathcal{B} reduces x to y holds $\mathcal{P}[x, y]$ provided the parameters have the following properties:

- For all elements x, y of \mathcal{A} such that $\langle x, y \rangle \in \mathcal{B}$ holds $\mathcal{P}[x, y]$,
- For every element x of \mathcal{A} holds $\mathcal{P}[x, x]$, and
- For all elements x, y, z of \mathcal{A} such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

We now state a number of propositions:

- (68) Let T be an antisymmetric non void reflexive transitive Noetherian TAS -structure with l.u.b.'s with structured adjectives and t_1, t_2 be types of T . If $\circ \rightarrow_T$ reduces t_1 to t_2 , then $t_1 \leq t_2$.
- (69) Let T be a Noetherian reflexive transitive antisymmetric non void TAS -structure with l.u.b.'s with structured adjectives. Then $\circ \rightarrow_T$ is irreflexive.
- (70) Let T be an antisymmetric non void reflexive transitive Noetherian TAS -structure with l.u.b.'s with structured adjectives. Then $\circ \rightarrow_T$ is strongly-normalizing.
- (71) Let T be a Noetherian reflexive transitive antisymmetric non void TAS -structure with l.u.b.'s with structured adjectives, t be a type of T , and A be a finite subset of the set of adjectives of T . Suppose that for every subset C of the set of adjectives of T such that $C \subseteq A$ and C is properly applicable to t and $A * t = C * t$ holds $C = A$. Let s be an one-to-one finite sequence of elements of the set of adjectives of T . Suppose $\text{rng } s = A$ and s is properly applicable to t . Let i be a natural number. If $1 \leq i$ and $i \leq \text{len } s$, then $\langle (\text{apply}(s, t))(i + 1), (\text{apply}(s, t))(i) \rangle \in \circ \rightarrow_T$.
- (72) Let T be a Noetherian reflexive transitive antisymmetric non void TAS -structure with l.u.b.'s with structured adjectives, t be a type of T , and A be a finite subset of the set of adjectives of T . Suppose that for every subset C of the set of adjectives of T such that $C \subseteq A$ and C is properly applicable to t and $A * t = C * t$ holds $C = A$. Let s be an one-to-one finite sequence of elements of the set of adjectives of T . Suppose $\text{rng } s = A$ and s is properly applicable to t . Then $\text{Rev}(\text{apply}(s, t))$ is a reduction sequence w.r.t. $\circ \rightarrow_T$.
- (73) Let T be a Noetherian reflexive transitive antisymmetric non void TAS -structure with l.u.b.'s with structured adjectives, t be a type of T , and A be a finite subset of the set of adjectives of T . If A is properly applicable to t , then $\circ \rightarrow_T$ reduces $A * t$ to t .
- (74) Let X be a non empty set, R be a binary relation on X , and r be a reduction sequence w.r.t. R . If $r(1) \in X$, then r is a finite sequence of elements of X .
- (75) Let X be a non empty set, R be a binary relation on X , x be an element of X , and y be a set. If R reduces x to y , then $y \in X$.
- (76) Let X be a non empty set and R be a binary relation on X . Suppose R is weakly-normalizing and has unique normal form property. Let x be an element of X . Then $\text{nf}_R(x) \in X$.
- (77) Let T be a Noetherian reflexive transitive antisymmetric non void TAS -structure with l.u.b.'s with structured adjectives and t_1, t_2 be types of T . Suppose $\circ \rightarrow_T$ reduces t_1 to t_2 . Then there exists a finite subset A of the set

of adjectives of T such that A is properly applicable to t_2 and $t_1 = A * t_2$.

- (78) Let T be an antisymmetric commutative non void reflexive transitive Noetherian TAS -structure with l.u.b.'s with structured adjectives. Then $\circ \rightarrow_T$ has Church-Rosser property and unique normal form property.

6. RADIX TYPES

Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian TAS -structure with structured adjectives and l.u.b.'s and let t be a type of T . The functor $\text{radix } t$ yielding a type of T is defined by:

(Def. 32) $\text{radix } t = \text{nf}_{\circ \rightarrow_T}(t)$.

We now state several propositions:

- (79) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian TAS -structure with structured adjectives and l.u.b.'s and t be a type of T . Then $\circ \rightarrow_T$ reduces t to $\text{radix } t$.
- (80) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian TAS -structure with structured adjectives and l.u.b.'s and t be a type of T . Then $t \leq \text{radix } t$.
- (81) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian TAS -structure with structured adjectives and l.u.b.'s, t be a type of T , and X be a set. Suppose $X = \{t'; t' \text{ ranges over types of } T: \bigvee_A: \text{finite subset of the set of adjectives of } T (A \text{ is properly applicable to } t' \wedge A * t' = t)\}$. Then $\text{sup } X$ exists in T and $\text{radix } t = \bigsqcup_T X$.
- (82) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian TAS -structure with structured adjectives and l.u.b.'s, t_1, t_2 be types of T , and a be an adjective of T . If a is properly applicable to t_1 and $a * t_1 \leq \text{radix } t_2$, then $t_1 \leq \text{radix } t_2$.
- (83) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian TAS -structure with structured adjectives and l.u.b.'s and t_1, t_2 be types of T . If $t_1 \leq t_2$, then $\text{radix } t_1 \leq \text{radix } t_2$.
- (84) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian TAS -structure with structured adjectives and l.u.b.'s, t be a type of T , and a be an adjective of T . If a is properly applicable to t , then $\text{radix}(a * t) = \text{radix } t$.

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