FORMALIZED MATHEMATICS
Volume 11, Number 4, 2003
University of Białystok

On Semilattice Structure of Mizar Types

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Summary. The aim of this paper is to develop a formal theory of Mizar types. The presented theory is an approach to the structure of Mizar types as a sup-semilattice with widening (subtyping) relation as the order. It is an abstraction from the existing implementation of the Mizar verifier and formalization of the ideas from [9].

MML Identifier: `ABCMIZ_0`.

The articles [20], [14], [24], [26], [23], [25], [3], [21], [1], [11], [12], [16], [10], [13], [18], [15], [4], [2], [19], [22], [5], [6], [7], [8], and [17] provide the terminology and notation for this paper.

1. Semilattice of Widening

Let us mention that every non empty relational structure which is trivial and reflexive is also complete.

Let T be a relational structure. A type of T is an element of T.

Let T be a relational structure. We say that T is Noetherian if and only if:

(Def. 1) The internal relation of T is reversely well founded.

Let us observe that every non empty relational structure which is trivial is also Noetherian.

Let T be a non empty relational structure. Let us observe that T is Noetherian if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let A be a non empty subset of T. Then there exists an element a of T such that $a \in A$ and for every element b of T such that $b \in A$ holds $a \not< b$.

Let T be a poset. We say that T is Mizar-widening-like if and only if:

(Def. 3) T is a sup-semilattice and Noetherian.

ISSN 1426–2630

Let us mention that every poset which is Mizar-widening-like is also Noetherian and upper-bounded and has l.u.b.'s.

Let us note that every sup-semilattice which is Noetherian is also Mizar-widening-like.

Let us observe that there exists a complete sup-semilattice which is Mizar-widening-like.

Let T be a Noetherian relational structure. One can check that the internal relation of T is reversely well founded.

Next we state the proposition

(1) For every Noetherian sup-semilattice T and for every ideal I of T holds sup I exists in T and $\sup I \in I$.

2. Adjectives

We consider adjective structures as systems

$\langle$ a set of adjectives, an operation non $\rangle$,

where the set of adjectives is a set and the operation non is a unary operation on the set of adjectives.

Let A be an adjective structure. We say that A is void if and only if:

(Def. 4) The set of adjectives of A is empty.

An adjective of A is an element of the set of adjectives of A.

The following proposition is true

(2) Let A_1, A_2 be adjective structures. Suppose the set of adjectives of $A_1 =$ the set of adjectives of A_2. If A_1 is void, then A_2 is void.

Let A be an adjective structure and let a be an element of the set of adjectives of A. The functor $\mathrm{non}\, a$ yields an adjective of A and is defined as follows:

(Def. 5) $\mathrm{non}\, a =$ (the operation non of A)(a).

One can prove the following proposition

(3) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2. Let a_1 be an adjective of A_1 and a_2 be an adjective of A_2. If $a_1 = a_2$, then $\mathrm{non}\, a_1 = \mathrm{non}\, a_2$.

Let A be an adjective structure. We say that A is involutive if and only if:

(Def. 6) For every adjective a of A holds $\mathrm{non}\,\mathrm{non}\, a = a$.

We say that A is without fixpoints if and only if:

(Def. 7) It is not true that there exists an adjective a of A such that $\mathrm{non}\, a = a$.

We now state three propositions:

(4) Let a_1, a_2 be sets. Suppose $a_1 \neq a_2$. Let A be an adjective structure. Suppose the set of adjectives of $A = \{a_1, a_2\}$ and (the operation non of A)$(a_1) = a_2$ and (the operation non of A)$(a_2) = a_1$. Then A is non void, involutive, and without fixpoints.

(5) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2. If A_1 is involutive, then A_2 is involutive.

(6) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2. If A_1 is without fixpoints, then A_2 is without fixpoints.

Let us observe that there exists a strict adjective structure which is non void, involutive, and without fixpoints.

Let A be a non void adjective structure. Observe that the set of adjectives of A is non empty.

We consider *TA*-structures as extensions of relational structure and adjective structure as systems

$\langle$ a carrier, a set of adjectives, an internal relation, an operation non, an adjective map $\rangle$,

where the carrier and the set of adjectives are sets, the internal relation is a binary relation on the carrier, the operation non is a unary operation on the set of adjectives, and the adjective map is a function from the carrier into Fin the set of adjectives.

Let X be a non empty set, let A be a set, let r be a binary relation on X, let n be a unary operation on A, and let a be a function from X into $\mathrm{Fin}\,A$. Observe that $\langle X, A, r, n, a\rangle$ is non empty.

Let X be a set, let A be a non empty set, let r be a binary relation on X, let n be a unary operation on A, and let a be a function from X into $\mathrm{Fin}\,A$. One can check that $\langle X, A, r, n, a\rangle$ is non void.

One can check that there exists a *TA*-structure which is trivial, reflexive, non empty, non void, involutive, without fixpoints, and strict.

Let T be a *TA*-structure and let t be an element of T. The functor $\mathrm{adjs}\,t$ yields a subset of the set of adjectives of T and is defined as follows:

(Def. 8) $\mathrm{adjs}\,t = (\text{the adjective map of } T)(t)$.

One can prove the following proposition

(7) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. Let t_1 be a type of T_1 and t_2 be a type of T_2. If $t_1 = t_2$, then $\mathrm{adjs}\,t_1 = \mathrm{adjs}\,t_2$.

Let T be a *TA*-structure. We say that T is consistent if and only if:

(Def. 9) For every type t of T and for every adjective a of T such that $a \in \mathrm{adjs}\,t$ holds $\mathrm{non}\,a \notin \mathrm{adjs}\,t$.

Next we state the proposition

(8) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. If T_1 is consistent, then T_2 is consistent.

Let T be a non empty TA-structure. We say that T has structured adjectives if and only if:

(Def. 10) The adjective map of T is a join-preserving map from T into $(2_{\subseteq}^{\text{the set of adjectives of } T})^{\mathrm{op}}$.

We now state the proposition

(9) Let T_1, T_2 be non empty TA-structures. Suppose the TA-structure of $T_1 =$ the TA-structure of T_2. If T_1 has structured adjectives, then T_2 has structured adjectives.

Let T be a reflexive transitive antisymmetric TA-structure with l.u.b.'s. Let us observe that T has structured adjectives if and only if:

(Def. 11) For all types t_1, t_2 of T holds $\mathrm{adjs}(t_1 \sqcup t_2) = \mathrm{adjs}\, t_1 \cap \mathrm{adjs}\, t_2$.

One can prove the following proposition

(10) Let T be a reflexive transitive antisymmetric TA-structure with l.u.b.'s. Suppose T has structured adjectives. Let t_1, t_2 be types of T. If $t_1 \leqslant t_2$, then $\mathrm{adjs}\, t_2 \subseteq \mathrm{adjs}\, t_1$.

Let T be a TA-structure and let a be an element of the set of adjectives of T. The functor $\mathrm{types}\, a$ yields a subset of T and is defined as follows:

(Def. 12) For every set x holds $x \in \mathrm{types}\, a$ iff there exists a type t of T such that $x = t$ and $a \in \mathrm{adjs}\, t$.

Let T be a non empty TA-structure and let A be a subset of the set of adjectives of T. The functor $\mathrm{types}\, A$ yielding a subset of T is defined as follows:

(Def. 13) For every type t of T holds $t \in \mathrm{types}\, A$ iff for every adjective a of T such that $a \in A$ holds $t \in \mathrm{types}\, a$.

One can prove the following propositions:

(11) Let T_1, T_2 be TA-structures. Suppose the TA-structure of $T_1 =$ the TA-structure of T_2. Let a_1 be an adjective of T_1 and a_2 be an adjective of T_2. If $a_1 = a_2$, then $\mathrm{types}\, a_1 = \mathrm{types}\, a_2$.

(12) For every non empty TA-structure T and for every adjective a of T holds $\mathrm{types}\, a = \{t;\, t \text{ ranges over types of } T\colon a \in \mathrm{adjs}\, t\}$.

(13) Let T be a TA-structure, t be a type of T, and a be an adjective of T. Then $a \in \mathrm{adjs}\, t$ if and only if $t \in \mathrm{types}\, a$.

(14) Let T be a non empty TA-structure, t be a type of T, and A be a subset of the set of adjectives of T. Then $A \subseteq \mathrm{adjs}\, t$ if and only if $t \in \mathrm{types}\, A$.

(15) For every non void TA-structure T and for every type t of T holds $\mathrm{adjs}\, t = \{a;\, a \text{ ranges over adjectives of } T\colon t \in \mathrm{types}\, a\}$.

(16) Let T be a non empty TA-structure and t be a type of T. Then $\mathrm{types}(\emptyset_{\text{the set of adjectives of } T}) =$ the carrier of T.

Let T be a TA-structure. We say that T has typed adjectives if and only if:

(Def. 14) For every adjective a of T holds $\mathrm{types}\, a \cup \mathrm{types}\, \mathrm{non}\, a$ is non empty.

We now state the proposition

(17) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of T_1 = the *TA*-structure of T_2. If T_1 has typed adjectives, then T_2 has typed adjectives.

Let us mention that there exists a complete upper-bounded non empty trivial reflexive transitive antisymmetric strict *TA*-structure which is non void, Mizar-widening-like, involutive, without fixpoints, and consistent and has structured adjectives and typed adjectives.

Next we state the proposition

(18) For every consistent *TA*-structure T and for every adjective a of T holds $\mathrm{types}\,a$ misses $\mathrm{types}\,\mathrm{non}\,a$.

Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives and let a be an adjective of T. Note that $\mathrm{types}\,a$ is lower and directed.

Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives and let A be a subset of the set of adjectives of T. One can verify that $\mathrm{types}\,A$ is lower and directed.

We now state the proposition

(19) Let T be reflexive antisymmetric transitive *TA*-structure with l.u.b.'s with structured adjectives and a be an adjective of T. Then $\mathrm{types}\,a$ is empty or $\mathrm{types}\,a$ is an ideal of T.

3. Applicability of Adjectives

Let T be a *TA*-structure, let t be an element of T, and let a be an adjective of T. We say that a is applicable to t if and only if:

(Def. 15) There exists a type t' of T such that $t' \in \mathrm{types}\,a$ and $t' \leqslant t$.

Let T be a *TA*-structure, let t be a type of T, and let A be a subset of the set of adjectives of T. We say that A is applicable to t if and only if:

(Def. 16) There exists a type t' of T such that $A \subseteq \mathrm{adjs}\,t'$ and $t' \leqslant t$.

We now state the proposition

(20) Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, a be an adjective of T, and t be a type of T. If a is applicable to t, then $\mathrm{types}\,a \cap {\downarrow}t$ is an ideal of T.

Let T be a non empty reflexive transitive *TA*-structure, let t be an element of T, and let a be an adjective of T. The functor $a * t$ yielding a type of T is defined by:

(Def. 17) $a * t = \sup(\mathrm{types}\,a \cap {\downarrow}t)$.

The following propositions are true:

(21) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If a is applicable to t, then $a * t \leqslant t$.

(22) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If a is applicable to t, then $a \in \mathrm{adjs}(a * t)$.

(23) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If a is applicable to t, then $a * t \in \mathrm{types}\, a$.

(24) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, a be an adjective of T, and t' be a type of T. If $t' \leqslant t$ and $a \in \mathrm{adjs}\, t'$, then a is applicable to t and $t' \leqslant a * t$.

(25) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If $a \in \mathrm{adjs}\, t$, then a is applicable to t and $a * t = t$.

(26) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a, b be adjectives of T. Suppose a is applicable to t and b is applicable to $a * t$. Then b is applicable to t and a is applicable to $b*t$ and $a*(b*t) = b*(a*t)$.

(27) Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, A be a subset of the set of adjectives of T, and t be a type of T. If A is applicable to t, then $\mathrm{types}\, A \cap {\downarrow}t$ is an ideal of T.

Let T be a non empty reflexive transitive *TA*-structure, let t be a type of T, and let A be a subset of the set of adjectives of T. The functor $A * t$ yielding a type of T is defined as follows:

(Def. 18) $A * t = \sup(\mathrm{types}\, A \cap {\downarrow}t)$.

Next we state the proposition

(28) Let T be a non empty reflexive transitive antisymmetric *TA*-structure and t be a type of T. Then $\emptyset_{\text{the set of adjectives of } T} * t = t$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let p be a finite sequence of elements of the set of adjectives of T. The functor $\mathrm{apply}(p,t)$ yielding a finite sequence of elements of the carrier of T is defined by the conditions (Def. 19).

(Def. 19)(i) $\mathrm{len}\, \mathrm{apply}(p,t) = \mathrm{len}\, p + 1$,

(ii) $(\mathrm{apply}(p,t))(1) = t$, and

(iii) for every natural number i and for every adjective a of T and for every type t of T such that $i \in \mathrm{dom}\, p$ and $a = p(i)$ and $t = (\mathrm{apply}(p,t))(i)$ holds $(\mathrm{apply}(p,t))(i+1) = a * t$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let p be a finite sequence of elements of the set of adjectives of T. Note that $\mathrm{apply}(p, t)$ is non empty.

One can prove the following two propositions:

(29) Let T be a non empty non void reflexive transitive *TA*-structure and t be a type of T. Then $\mathrm{apply}(\varepsilon_{(\text{the set of adjectives of } T)}, t) = \langle t \rangle$.

(30) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and a be an adjective of T. Then $\mathrm{apply}(\langle a \rangle, t) = \langle t, a * t \rangle$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let v be a finite sequence of elements of the set of adjectives of T. The functor $v * t$ yielding a type of T is defined by:

(Def. 20) $v * t = (\mathrm{apply}(v, t))(\mathrm{len}\, v + 1)$.

The following propositions are true:

(31) Let T be a non empty non void reflexive transitive *TA*-structure and t be a type of T. Then $\varepsilon_{(\text{the set of adjectives of } T)} * t = t$.

(32) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and a be an adjective of T. Then $\langle a \rangle * t = a * t$.

(33) For all finite sequences p, q and for every natural number i such that $i \geqslant 1$ and $i < \mathrm{len}\, p$ holds $(p \,{}^{\$\frown}\, q)(i) = p(i)$.

(34) Let p be a non empty finite sequence, q be a finite sequence, and i be a natural number. If $i < \mathrm{len}\, q$, then $(p \,{}^{\$\frown}\, q)(\mathrm{len}\, p + i) = q(i + 1)$.

(35) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Then $\mathrm{apply}(v_1 \frown v_2, t) = (\mathrm{apply}(v_1, t)) \,{}^{\$\frown}\, \mathrm{apply}(v_2, v_1 * t)$.

(36) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, v_1, v_2 be finite sequences of elements of the set of adjectives of T, and i be a natural number. If $i \in \mathrm{dom}\, v_1$, then $(\mathrm{apply}(v_1 \frown v_2, t))(i) = (\mathrm{apply}(v_1, t))(i)$.

(37) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Then $(\mathrm{apply}(v_1 \frown v_2, t))(\mathrm{len}\, v_1 + 1) = v_1 * t$.

(38) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Then $v_2 * (v_1 * t) = (v_1 \frown v_2) * t$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let v be a finite sequence of elements of the set of adjectives of T. We say that v is applicable to t if and only if the condition (Def. 21) is satisfied.

(Def. 21) Let i be a natural number, a be an adjective of T, and s be a type of T. If $i \in \mathrm{dom}\, v$ and $a = v(i)$ and $s = (\mathrm{apply}(v, t))(i)$, then a is applicable to s.

Next we state a number of propositions:

(39) Let T be a non empty non void reflexive transitive *TA*-structure and t be a type of T. Then $\varepsilon_{(\text{the set of adjectives of } T)}$ is applicable to t.

(40) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and a be an adjective of T. Then a is applicable to t if and only if $\langle a \rangle$ is applicable to t.

(41) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose $v_1 \frown v_2$ is applicable to t. Then v_1 is applicable to t and v_2 is applicable to $v_1 * t$.

(42) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let i_1, i_2 be natural numbers. Suppose $1 \leqslant i_1$ and $i_1 \leqslant i_2$ and $i_2 \leqslant \operatorname{len} v + 1$. Let t_1, t_2 be types of T. If $t_1 = (\operatorname{apply}(v, t))(i_1)$ and $t_2 = (\operatorname{apply}(v, t))(i_2)$, then $t_2 \leqslant t_1$.

(43) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let s be a type of T. If $s \in \operatorname{rng} \operatorname{apply}(v, t)$, then $v * t \leqslant s$ and $s \leqslant t$.

(44) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. If v is applicable to t, then $v * t \leqslant t$.

(45) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. If v is applicable to t, then $\operatorname{rng} v \subseteq \operatorname{adjs}(v * t)$.

(46) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let A be a subset of the set of adjectives of T. If $A = \operatorname{rng} v$, then A is applicable to t.

(47) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose v_1 is applicable to t and $\operatorname{rng} v_2 \subseteq \operatorname{rng} v_1$. Let s be a type of T. If $s \in \operatorname{rng} \operatorname{apply}(v_2, t)$, then $v_1 * t \leqslant s$.

(48) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-

structure with l.u.b.'s with structured adjectives, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. If $v_1 \frown v_2$ is applicable to t, then $v_2 \frown v_1$ is applicable to t.

(49) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. If $v_1 \frown v_2$ is applicable to t, then $(v_1 \frown v_2) * t = (v_2 \frown v_1) * t$.

(50) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If A is applicable to t, then $A * t \leqslant t$.

(51) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If A is applicable to t, then $A \subseteq \mathrm{adjs}(A * t)$.

(52) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If A is applicable to t, then $A * t \in \mathrm{types}\, A$.

(53) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, A be a subset of the set of adjectives of T, and t' be a type of T. If $t' \leqslant t$ and $A \subseteq \mathrm{adjs}\, t'$, then A is applicable to t and $t' \leqslant A * t$.

(54) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If $A \subseteq \mathrm{adjs}\, t$, then A is applicable to t and $A * t = t$.

(55) Let T be a *TA*-structure, t be a type of T, and A, B be subsets of the set of adjectives of T. If A is applicable to t and $B \subseteq A$, then B is applicable to t.

(56) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, a be an adjective of T, and A, B be subsets of the set of adjectives of T. If $B = A \cup \{a\}$ and B is applicable to t, then $a * (A * t) = B * t$.

(57) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let A be a subset of the set of adjectives of T. If $A = \mathrm{rng}\, v$, then $v * t = A * t$.

4. Subject Function

Let T be a non empty non void *TA*-structure. The functor $\operatorname{sub} T$ yields a function from the set of adjectives of T into the carrier of T and is defined as follows:

(Def. 22) For every adjective a of T holds $(\operatorname{sub} T)(a) = \sup(\operatorname{types} a \cup \operatorname{types} \operatorname{non} a)$.

We introduce *TAS*-structures which are extensions of *TA*-structure and are systems

⟨ a carrier, a set of adjectives, an internal relation, an operation non, an adjective map, a subject map ⟩,

where the carrier and the set of adjectives are sets, the internal relation is a binary relation on the carrier, the operation non is a unary operation on the set of adjectives, the adjective map is a function from the carrier into Fin the set of adjectives, and the subject map is a function from the set of adjectives into the carrier.

Let us observe that there exists a *TAS*-structure which is non void, reflexive, trivial, non empty, and strict.

Let T be a non empty non void *TAS*-structure and let a be an adjective of T. The functor $\operatorname{sub} a$ yields a type of T and is defined as follows:

(Def. 23) $\operatorname{sub} a =$ (the subject map of T)(a).

Let T be a non empty non void *TAS*-structure. We say that T is absorbing non if and only if:

(Def. 24) (The subject map of T) $\cdot$ (the operation non of T) = the subject map of T.

We say that T is subjected if and only if:

(Def. 25) For every adjective a of T holds $\operatorname{types} a \cup \operatorname{types} \operatorname{non} a \leqslant \operatorname{sub} a$ and if $\operatorname{types} a \neq \emptyset$ and $\operatorname{types} \operatorname{non} a \neq \emptyset$, then $\operatorname{sub} a = \sup(\operatorname{types} a \cup \operatorname{types} \operatorname{non} a)$.

Let T be a non empty non void *TAS*-structure. Let us observe that T is absorbing non if and only if:

(Def. 26) For every adjective a of T holds $\operatorname{sub} \operatorname{non} a = \operatorname{sub} a$.

Let T be a non empty non void *TAS*-structure, let t be an element of T, and let a be an adjective of T. We say that a is properly applicable to t if and only if:

(Def. 27) $t \leqslant \operatorname{sub} a$ and a is applicable to t.

Let T be a non empty non void reflexive transitive *TAS*-structure, let t be a type of T, and let v be a finite sequence of elements of the set of adjectives of T. We say that v is properly applicable to t if and only if the condition (Def. 28) is satisfied.

(Def. 28) Let i be a natural number, a be an adjective of T, and s be a type of T. If $i \in \operatorname{dom} v$ and $a = v(i)$ and $s = (\operatorname{apply}(v, t))(i)$, then a is properly applicable to s.

One can prove the following propositions:

(58) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. If v is properly applicable to t, then v is applicable to t.

(59) Let T be a non empty non void reflexive transitive *TAS*-structure and t be a type of T. Then $\varepsilon_{(\text{the set of adjectives of } T)}$ is properly applicable to t.

(60) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and a be an adjective of T. Then a is properly applicable to t if and only if $\langle a \rangle$ is properly applicable to t.

(61) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose $v_1 \frown v_2$ is properly applicable to t. Then v_1 is properly applicable to t and v_2 is properly applicable to $v_1 * t$.

(62) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose v_1 is properly applicable to t and v_2 is properly applicable to $v_1 * t$. Then $v_1 \frown v_2$ is properly applicable to t.

Let T be a non empty non void reflexive transitive *TAS*-structure, let t be a type of T, and let A be a subset of the set of adjectives of T. We say that A is properly applicable to t if and only if the condition (Def. 29) is satisfied.

(Def. 29) There exists a finite sequence s of elements of the set of adjectives of T such that $\operatorname{rng} s = A$ and s is properly applicable to t.

Next we state two propositions:

(63) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and A be a subset of the set of adjectives of T. If A is properly applicable to t, then A is finite.

(64) Let T be a non empty non void reflexive transitive *TAS*-structure and t be a type of T. Then $\emptyset_{\text{the set of adjectives of } T}$ is properly applicable to t.

The scheme *MinimalFiniteSet* concerns a unary predicate $\mathcal{P}$, and states that:

There exists a finite set A such that $\mathcal{P}[A]$ and for every set B such that $B \subseteq A$ and $\mathcal{P}[B]$ holds $B = A$

provided the following requirement is met:

- There exists a finite set A such that $\mathcal{P}[A]$.

One can prove the following proposition

(65) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and A be a subset of the set of adjectives of T. Suppose A is properly applicable to t. Then there exists a subset B of the set of adjectives of T such that

(i) $B \subseteq A$,

(ii) B is properly applicable to t,

(iii) $A * t = B * t$, and

(iv) for every subset C of the set of adjectives of T such that $C \subseteq B$ and C is properly applicable to t and $A * t = C * t$ holds $C = B$.

Let T be a non empty non void reflexive transitive *TAS*-structure. We say that T is commutative if and only if the condition (Def. 30) is satisfied.

(Def. 30) Let t_1, t_2 be types of T and a be an adjective of T. Suppose a is properly applicable to t_1 and $a * t_1 \leqslant t_2$. Then there exists a finite subset A of the set of adjectives of T such that A is properly applicable to $t_1 \sqcup t_2$ and $A * (t_1 \sqcup t_2) = t_2$.

Let us observe that there exists a complete upper-bounded non empty non void trivial reflexive transitive antisymmetric strict *TAS*-structure which is Mizar-widening-like, involutive, without fixpoints, consistent, absorbing non, subjected, and commutative and has structured adjectives and typed adjectives.

Next we state the proposition

(66) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. Suppose A is properly applicable to t. Then there exists an one-to-one finite sequence s of elements of the set of adjectives of T such that $\operatorname{rng} s = A$ and s is properly applicable to t.

5. Reduction of Adjectives

Let T be a non empty non void reflexive transitive *TAS*-structure. The functor $\circ\!\!\rightarrow_T$ yields a binary relation on T and is defined by the condition (Def. 31).

(Def. 31) Let t_1, t_2 be types of T. Then $\langle t_1, t_2 \rangle \in \circ\!\!\rightarrow_T$ if and only if there exists an adjective a of T such that $a \notin \operatorname{adjs} t_2$ and a is properly applicable to t_2 and $a * t_2 = t_1$.

Next we state the proposition

(67) Let T be an antisymmetric non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\rightarrow_T \subseteq$ the internal relation of T.

The scheme *RedInd* deals with a non empty set $\mathcal{A}$, a binary relation $\mathcal{B}$ on $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

For all elements x, y of $\mathcal{A}$ such that $\mathcal{B}$ reduces x to y holds $\mathcal{P}[x, y]$

provided the parameters have the following properties:

- For all elements x, y of $\mathcal{A}$ such that $\langle x, y \rangle \in \mathcal{B}$ holds $\mathcal{P}[x, y]$,
- For every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$, and
- For all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

We now state a number of propositions:

(68) Let T be an antisymmetric non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives and t_1, t_2 be types of T. If $\circ\!\!\rightarrow_T$ reduces t_1 to t_2, then $t_1 \leqslant t_2$.

(69) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\rightarrow_T$ is irreflexive.

(70) Let T be an antisymmetric non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\rightarrow_T$ is strongly-normalizing.

(71) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a finite subset of the set of adjectives of T. Suppose that for every subset C of the set of adjectives of T such that $C \subseteq A$ and C is properly applicable to t and $A * t = C * t$ holds $C = A$. Let s be an one-to-one finite sequence of elements of the set of adjectives of T. Suppose $\operatorname{rng} s = A$ and s is properly applicable to t. Let i be a natural number. If $1 \leqslant i$ and $i \leqslant \operatorname{len} s$, then $\langle(\operatorname{apply}(s,t))(i+1),\ (\operatorname{apply}(s,t))(i)\rangle \in \circ\!\!\rightarrow_T$.

(72) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a finite subset of the set of adjectives of T. Suppose that for every subset C of the set of adjectives of T such that $C \subseteq A$ and C is properly applicable to t and $A * t = C * t$ holds $C = A$. Let s be an one-to-one finite sequence of elements of the set of adjectives of T. Suppose $\operatorname{rng} s = A$ and s is properly applicable to t. Then $\operatorname{Rev}(\operatorname{apply}(s,t))$ is a reduction sequence w.r.t. $\circ\!\!\rightarrow_T$.

(73) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a finite subset of the set of adjectives of T. If A is properly applicable to t, then $\circ\!\!\rightarrow_T$ reduces $A * t$ to t.

(74) Let X be a non empty set, R be a binary relation on X, and r be a reduction sequence w.r.t. R. If $r(1) \in X$, then r is a finite sequence of elements of X.

(75) Let X be a non empty set, R be a binary relation on X, x be an element of X, and y be a set. If R reduces x to y, then $y \in X$.

(76) Let X be a non empty set and R be a binary relation on X. Suppose R is weakly-normalizing and has unique normal form property. Let x be an element of X. Then $\operatorname{nf}_R(x) \in X$.

(77) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives and t_1, t_2 be types of T. Suppose $\circ\!\!\rightarrow_T$ reduces t_1 to t_2. Then there exists a finite subset A of the set

of adjectives of T such that A is properly applicable to t_2 and $t_1 = A * t_2$.

(78) Let T be an antisymmetric commutative non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\rightarrow_T$ has Church-Rosser property and unique normal form property.

6. Radix Types

Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and let t be a type of T. The functor $\operatorname{radix} t$ yielding a type of T is defined by:

(Def. 32) $\operatorname{radix} t = \mathrm{nf}_{\circ\!\rightarrow_T}(t)$.

We now state several propositions:

(79) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and t be a type of T. Then $\circ\!\!\rightarrow_T$ reduces t to $\operatorname{radix} t$.

(80) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and t be a type of T. Then $t \leqslant \operatorname{radix} t$.

(81) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s, t be a type of T, and X be a set. Suppose $X = \{t'; t'$ ranges over types of T: $\bigvee_{A:\ \text{finite subset of the set of adjectives of } T}$ (A is properly applicable to t' $\wedge$ $A * t' = t)\}$. Then sup X exists in T and $\operatorname{radix} t = \bigsqcup_T X$.

(82) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s, t_1, t_2 be types of T, and a be an adjective of T. If a is properly applicable to t_1 and $a * t_1 \leqslant \operatorname{radix} t_2$, then $t_1 \leqslant \operatorname{radix} t_2$.

(83) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and t_1, t_2 be types of T. If $t_1 \leqslant t_2$, then $\operatorname{radix} t_1 \leqslant \operatorname{radix} t_2$.

(84) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s, t be a type of T, and a be an adjective of T. If a is properly applicable to t, then $\operatorname{radix}(a * t) = \operatorname{radix} t$.

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Received August 8, 2003