

Cross Products and Tripple Vector Products in 3-dimensional Euclidean Space

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Summary. First, we extend the basic theorems of 3-dimensional Euclidean space, and then define the cross product in the same space and relative vector relations using the above definition.

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The articles [14], [2], [12], [9], [6], [4], [3], [5], [13], [10], [11], [7], [8], and [1] provide the terminology and notation for this paper.

We adopt the following convention: x, y, z denote real numbers, x_3, y_3 denote elements of \mathbb{R} , and p denotes a point of \mathcal{E}_T^3 .

We now state the proposition

(1) There exist x, y, z such that $p = \langle x, y, z \rangle$.

Let us consider p . The functor p_1 yielding a real number is defined as follows:

(Def. 1) For every finite sequence f such that $p = f$ holds $p_1 = f(1)$.

The functor p_2 yields a real number and is defined by:

(Def. 2) For every finite sequence f such that $p = f$ holds $p_2 = f(2)$.

The functor p_3 yields a real number and is defined by:

(Def. 3) For every finite sequence f such that $p = f$ holds $p_3 = f(3)$.

Let us consider x, y, z . The functor $[x, y, z]$ yields a point of \mathcal{E}_T^3 and is defined as follows:

(Def. 4) $[x, y, z] = \langle x, y, z \rangle$.

One can prove the following three propositions:

(2) $[x, y, z]_1 = x$ and $[x, y, z]_2 = y$ and $[x, y, z]_3 = z$.

(3) $p = [p_1, p_2, p_3]$.

$$(4) \quad 0_{\mathcal{E}_T^3} = [0, 0, 0].$$

We adopt the following rules: p_1, p_2, p_3, p_4 are points of \mathcal{E}_T^3 and $x_1, x_2, y_1, y_2, z_1, z_2$ are real numbers.

Next we state several propositions:

$$(5) \quad p_1 + p_2 = [(p_1)_1 + (p_2)_1, (p_1)_2 + (p_2)_2, (p_1)_3 + (p_2)_3].$$

$$(6) \quad [x_1, y_1, z_1] + [x_2, y_2, z_2] = [x_1 + x_2, y_1 + y_2, z_1 + z_2].$$

$$(7) \quad x \cdot p = [x \cdot p_1, x \cdot p_2, x \cdot p_3].$$

$$(8) \quad x \cdot [x_1, y_1, z_1] = [x \cdot x_1, x \cdot y_1, x \cdot z_1].$$

$$(9) \quad (x \cdot p)_1 = x \cdot p_1 \text{ and } (x \cdot p)_2 = x \cdot p_2 \text{ and } (x \cdot p)_3 = x \cdot p_3.$$

$$(10) \quad -p = [-p_1, -p_2, -p_3].$$

$$(11) \quad -[x_1, y_1, z_1] = [-x_1, -y_1, -z_1].$$

$$(12) \quad p_1 - p_2 = [(p_1)_1 - (p_2)_1, (p_1)_2 - (p_2)_2, (p_1)_3 - (p_2)_3].$$

$$(13) \quad [x_1, y_1, z_1] - [x_2, y_2, z_2] = [x_1 - x_2, y_1 - y_2, z_1 - z_2].$$

Let us consider p_1, p_2 . The functor $p_1 \times p_2$ yielding a point of \mathcal{E}_T^3 is defined by:

$$(Def. 5) \quad p_1 \times p_2 = [(p_1)_2 \cdot (p_2)_3 - (p_1)_3 \cdot (p_2)_2, (p_1)_3 \cdot (p_2)_1 - (p_1)_1 \cdot (p_2)_3, (p_1)_1 \cdot (p_2)_2 - (p_1)_2 \cdot (p_2)_1].$$

The following propositions are true:

$$(14) \quad \text{If } p = [x, y, z], \text{ then } p_1 = x \text{ and } p_2 = y \text{ and } p_3 = z.$$

$$(15) \quad [x_1, y_1, z_1] \times [x_2, y_2, z_2] = [y_1 \cdot z_2 - z_1 \cdot y_2, z_1 \cdot x_2 - x_1 \cdot z_2, x_1 \cdot y_2 - y_1 \cdot x_2].$$

$$(16) \quad (x \cdot p_1) \times p_2 = x \cdot (p_1 \times p_2) \text{ and } (x \cdot p_1) \times p_2 = p_1 \times (x \cdot p_2).$$

$$(17) \quad p_1 \times p_2 = -p_2 \times p_1.$$

$$(18) \quad (-p_1) \times p_2 = p_1 \times -p_2.$$

$$(19) \quad [0, 0, 0] \times [x, y, z] = 0_{\mathcal{E}_T^3}.$$

$$(20) \quad [x_1, 0, 0] \times [x_2, 0, 0] = 0_{\mathcal{E}_T^3}.$$

$$(21) \quad [0, y_1, 0] \times [0, y_2, 0] = 0_{\mathcal{E}_T^3}.$$

$$(22) \quad [0, 0, z_1] \times [0, 0, z_2] = 0_{\mathcal{E}_T^3}.$$

$$(23) \quad p_1 \times (p_2 + p_3) = p_1 \times p_2 + p_1 \times p_3.$$

$$(24) \quad (p_1 + p_2) \times p_3 = p_1 \times p_3 + p_2 \times p_3.$$

$$(25) \quad p_1 \times p_1 = 0_{\mathcal{E}_T^3}.$$

$$(26) \quad (p_1 + p_2) \times (p_3 + p_4) = p_1 \times p_3 + p_1 \times p_4 + p_2 \times p_3 + p_2 \times p_4.$$

$$(27) \quad p = \langle p_1, p_2, p_3 \rangle.$$

$$(28) \quad \text{For all finite sequences } f_1, f_2 \text{ of elements of } \mathbb{R} \text{ such that } \text{len } f_1 = 3 \text{ and } \text{len } f_2 = 3 \text{ holds } f_1 \bullet f_2 = \langle f_1(1) \cdot f_2(1), f_1(2) \cdot f_2(2), f_1(3) \cdot f_2(3) \rangle.$$

$$(29) \quad |(p_1, p_2)| = (p_1)_1 \cdot (p_2)_1 + (p_1)_2 \cdot (p_2)_2 + (p_1)_3 \cdot (p_2)_3.$$

$$(30) \quad |([x_1, x_2, x_3], [y_1, y_2, y_3])| = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3.$$

Let us consider p_1, p_2, p_3 . The functor $\langle |p_1, p_2, p_3| \rangle$ yielding a real number is defined as follows:

$$\text{(Def. 6)} \quad \langle |p_1, p_2, p_3| \rangle = |(p_1, p_2 \times p_3)|.$$

The following propositions are true:

$$(31) \quad \langle |p_1, p_1, p_2| \rangle = 0 \text{ and } \langle |p_2, p_1, p_2| \rangle = 0.$$

$$(32) \quad p_1 \times (p_2 \times p_3) = |(p_1, p_3)| \cdot p_2 - |(p_1, p_2)| \cdot p_3.$$

$$(33) \quad \langle |p_1, p_2, p_3| \rangle = \langle |p_2, p_3, p_1| \rangle.$$

$$(34) \quad \langle |p_1, p_2, p_3| \rangle = \langle |p_3, p_1, p_2| \rangle.$$

$$(35) \quad \langle |p_1, p_2, p_3| \rangle = |(p_1 \times p_2, p_3)|.$$

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