

On the Upper and Lower Approximations of the Curve¹

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The papers [28], [32], [2], [15], [1], [5], [6], [4], [31], [16], [29], [17], [27], [13], [3], [25], [26], [10], [11], [8], [30], [14], [20], [18], [12], [23], [22], [24], [7], [9], [19], and [21] provide the terminology and notation for this paper.

In this paper n denotes a natural number.

Let C be a simple closed curve. The functor $\text{UpperAppr}(C)$ yields a sequence of subsets of the carrier of \mathcal{E}_T^2 and is defined as follows:

(Def. 1) For every natural number i holds $(\text{UpperAppr}(C))(i) = \text{UpperArc}(\tilde{\mathcal{L}}(\text{Cage}(C, i)))$.

The functor $\text{LowerAppr}(C)$ yielding a sequence of subsets of the carrier of \mathcal{E}_T^2 is defined as follows:

(Def. 2) For every natural number i holds $(\text{LowerAppr}(C))(i) = \text{LowerArc}(\tilde{\mathcal{L}}(\text{Cage}(C, i)))$.

Let C be a simple closed curve. The functor $\text{NorthArc}(C)$ yields a subset of \mathcal{E}_T^2 and is defined by:

(Def. 3) $\text{NorthArc}(C) = \text{Li } \text{UpperAppr}(C)$.

The functor $\text{SouthArc}(C)$ yielding a subset of \mathcal{E}_T^2 is defined as follows:

(Def. 4) $\text{SouthArc}(C) = \text{Li } \text{LowerAppr}(C)$.

We now state a number of propositions:

- (1) For all natural numbers n, m such that $n \leq m$ and $n \neq 0$ holds $\frac{n+1}{n} \geq \frac{m+1}{m}$.

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- (2) Let E be a compact non vertical non horizontal subset of \mathcal{E}_T^2 and m, j be natural numbers. Suppose $1 \leq m$ and $m \leq n$ and $1 \leq j$ and $j \leq \text{width Gauge}(E, n)$. Then $\mathcal{L}(\text{Gauge}(E, n) \circ (\text{Center Gauge}(E, n), \text{width Gauge}(E, n)), \text{Gauge}(E, n) \circ (\text{Center Gauge}(E, n), j)) \subseteq \mathcal{L}(\text{Gauge}(E, m) \circ (\text{Center Gauge}(E, m), \text{width Gauge}(E, m)), \text{Gauge}(E, n) \circ (\text{Center Gauge}(E, n), j))$.
- (3) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and i, j be natural numbers. Suppose $1 \leq i$ and $i \leq \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{Cage}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, \text{width Gauge}(C, n)), \text{Gauge}(C, n) \circ (i, j))$ meets $\tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$.
- (4) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and n be a natural number. Suppose $n > 0$. Let i, j be natural numbers. Suppose $1 \leq i$ and $i \leq \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{Cage}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, \text{width Gauge}(C, n)), \text{Gauge}(C, n) \circ (i, j))$ meets $\text{UpperArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n)))$.
- (5) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 and j be a natural number. Suppose $\text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j) \in \text{LowerArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n+1)))$ and $1 \leq j$ and $j \leq \text{width Gauge}(C, n+1)$. Then $\mathcal{L}(\text{Gauge}(C, 1) \circ (\text{Center Gauge}(C, 1), \text{width Gauge}(C, 1)), \text{Gauge}(C, n+1) \circ (\text{Center Gauge}(C, n+1), j))$ meets $\text{UpperArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n+1)))$.
- (6) Let C be a compact connected non vertical non horizontal subset of \mathcal{E}_T^2 , f be a finite sequence of elements of \mathcal{E}_T^2 , and k be a natural number. Suppose $1 \leq k$ and $k+1 \leq \text{len } f$ and f is a sequence which elements belong to $\text{Gauge}(C, n)$. Then $\rho(f_k, f_{k+1}) = \frac{\text{N-bound}(C) - \text{S-bound}(C)}{2^n}$ or $\rho(f_k, f_{k+1}) = \frac{\text{E-bound}(C) - \text{W-bound}(C)}{2^n}$.
- (7) Let M be a symmetric triangle metric structure, r be a real number, and p, q, x be elements of M . If $p \in \text{Ball}(x, r)$ and $q \in \text{Ball}(x, r)$, then $\rho(p, q) < 2 \cdot r$.
- (8) Let A be a subset of \mathcal{E}_T^n , p be a point of \mathcal{E}_T^n , and p' be a point of \mathcal{E}^n . Suppose $p = p'$. Let s be a real number. Suppose $s > 0$. Then $p \in \bar{A}$ if and only if for every real number r such that $0 < r$ and $r < s$ holds $\text{Ball}(p', r)$ meets A .
- (9) For every compact connected non vertical non horizontal subset C of \mathcal{E}_T^2 holds $\text{N-bound}(C) < \text{N-bound}(\tilde{\mathcal{L}}(\text{Cage}(C, n)))$.
- (10) For every compact connected non vertical non horizontal subset C of \mathcal{E}_T^2 holds $\text{E-bound}(C) < \text{E-bound}(\tilde{\mathcal{L}}(\text{Cage}(C, n)))$.
- (11) For every compact connected non vertical non horizontal subset C of \mathcal{E}_T^2

- holds $S\text{-bound}(\tilde{\mathcal{L}}(\text{Cage}(C, n))) < S\text{-bound}(C)$.
- (12) For every compact connected non vertical non horizontal subset C of \mathcal{E}_T^2 holds $W\text{-bound}(\tilde{\mathcal{L}}(\text{Cage}(C, n))) < W\text{-bound}(C)$.
- (13) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq k$ and $k \leq j$ and $j \leq \text{width Gauge}(C, n)$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, k), \text{Gauge}(C, n) \circ (i, j)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, k)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, k), \text{Gauge}(C, n) \circ (i, j)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, j)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, k), \text{Gauge}(C, n) \circ (i, j))$ meets $\text{UpperArc}(C)$.
- (14) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq k$ and $k \leq j$ and $j \leq \text{width Gauge}(C, n)$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, k), \text{Gauge}(C, n) \circ (i, j)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, k)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, k), \text{Gauge}(C, n) \circ (i, j)) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i, j)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, k), \text{Gauge}(C, n) \circ (i, j))$ meets $\text{LowerArc}(C)$.
- (15) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $n > 0$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k)) \cap \text{LowerArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n))) = \{\text{Gauge}(C, n) \circ (i, k)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k)) \cap \text{UpperArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n))) = \{\text{Gauge}(C, n) \circ (i, j)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k))$ meets $\text{UpperArc}(C)$.
- (16) Let C be a simple closed curve and i, j, k be natural numbers. Suppose that $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $n > 0$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k)) \cap \text{LowerArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n))) = \{\text{Gauge}(C, n) \circ (i, k)\}$ and $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k)) \cap \text{UpperArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n))) = \{\text{Gauge}(C, n) \circ (i, j)\}$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k))$ meets $\text{LowerArc}(C)$.
- (17) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, k) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k))$ meets $\text{UpperArc}(C)$.
- (18) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n)$ and $\text{Gauge}(C, n) \circ (i, k) \in \tilde{\mathcal{L}}(\text{LowerSeq}(C, n))$ and $\text{Gauge}(C, n) \circ (i, j) \in \tilde{\mathcal{L}}(\text{UpperSeq}(C, n))$. Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i, j), \text{Gauge}(C, n) \circ (i, k))$ meets $\text{LowerArc}(C)$.
- (19) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len Gauge}(C, n)$ and $1 \leq j$

- $(i_1, k), \text{Gauge}(C, n) \circ (i_2, k)) \cap \tilde{\mathcal{L}}(\text{UpperSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_1, j)\}$
 and $(\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))) \cap \tilde{\mathcal{L}}(\text{LowerSeq}(C, n)) = \{\text{Gauge}(C, n) \circ (i_2, k)\}$.
 Then $\mathcal{L}(\text{Gauge}(C, n) \circ (i_1, j), \text{Gauge}(C, n) \circ (i_1, k)) \cup \mathcal{L}(\text{Gauge}(C, n) \circ (i_1, k), \text{Gauge}(C, n) \circ (i_2, k))$ meets $\text{LowerArc}(C)$.
- (25) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \text{len Gauge}(C, n+1)$ and $1 < i_2$ and $i_2 < \text{len Gauge}(C, n+1)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (i_1, k) \in \text{LowerArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n+1)))$ and $\text{Gauge}(C, n+1) \circ (i_2, j) \in \text{UpperArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n+1)))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (i_2, j), \text{Gauge}(C, n+1) \circ (i_2, k)) \cup \mathcal{L}(\text{Gauge}(C, n+1) \circ (i_2, k), \text{Gauge}(C, n+1) \circ (i_1, k))$ meets $\text{LowerArc}(C)$.
- (26) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \text{len Gauge}(C, n+1)$ and $1 < i_2$ and $i_2 < \text{len Gauge}(C, n+1)$ and $1 \leq j$ and $j \leq k$ and $k \leq \text{width Gauge}(C, n+1)$ and $\text{Gauge}(C, n+1) \circ (i_1, k) \in \text{LowerArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n+1)))$ and $\text{Gauge}(C, n+1) \circ (i_2, j) \in \text{UpperArc}(\tilde{\mathcal{L}}(\text{Cage}(C, n+1)))$. Then $\mathcal{L}(\text{Gauge}(C, n+1) \circ (i_2, j), \text{Gauge}(C, n+1) \circ (i_2, k)) \cup \mathcal{L}(\text{Gauge}(C, n+1) \circ (i_2, k), \text{Gauge}(C, n+1) \circ (i_1, k))$ meets $\text{UpperArc}(C)$.
- (27) For every simple closed curve C and for every point p of \mathcal{E}_T^2 such that $\text{W-bound}(C) < p_1$ and $p_1 < \text{E-bound}(C)$ holds $p \notin \text{NorthArc}(C)$ or $p \notin \text{SouthArc}(C)$.
- (28) For every simple closed curve C and for every point p of \mathcal{E}_T^2 such that $p_1 = \frac{\text{W-bound}(C) + \text{E-bound}(C)}{2}$ holds $p \notin \text{NorthArc}(C)$ or $p \notin \text{SouthArc}(C)$.

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