

On the Kuratowski Limit Operators¹

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Summary. In the paper we give formal descriptions of the two Kuratowski limit operators: $\text{Li } S$ and $\text{Ls } S$, where S is an arbitrary sequence of subsets of a fixed topological space. In the two last sections we prove basic properties of these lower and upper topological limits, which may be found e.g. in [19]. In the sections 2–4, we present three operators which are associated in some sense with the above mentioned, that is $\liminf F$, $\limsup F$, and $\text{limes } F$, where F is a sequence of subsets of a fixed 1-sorted structure.

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The articles [30], [33], [2], [29], [9], [1], [22], [24], [35], [12], [34], [6], [4], [18], [8], [7], [16], [5], [13], [25], [31], [21], [10], [23], [14], [15], [20], [17], [27], [28], [26], [11], [3], and [32] provide the notation and terminology for this paper.

1. PRELIMINARIES

One can prove the following four propositions:

- (1) For all sets X , x and for every subset A of X such that $x \notin A$ and $x \in X$ holds $x \in A^c$.
- (2) For every function F and for every set i such that $i \in \text{dom } F$ holds $\bigcap F \subseteq F(i)$.
- (3) Let T be a non empty 1-sorted structure and S_1, S_2 be sequences of subsets of the carrier of T . Then $S_1 = S_2$ if and only if for every natural number n holds $S_1(n) = S_2(n)$.
- (4) For all sets A, B, C, D such that A meets B and C meets D holds $\{A, C\}$ meets $\{B, D\}$.

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Let X be a 1-sorted structure. Note that every sequence of subsets of the carrier of X is non empty.

Let T be a non empty 1-sorted structure. One can check that there exists a sequence of subsets of the carrier of T which is non-empty.

Let T be a non empty 1-sorted structure.

(Def. 1) A sequence of subsets of the carrier of T is said to be a sequence of subsets of T .

In this article we present several logical schemes. The scheme *LambdaSSeq* deals with a non empty 1-sorted structure \mathcal{A} and a unary functor \mathcal{F} yielding a subset of \mathcal{A} , and states that:

There exists a sequence f of subsets of \mathcal{A} such that for every natural number n holds $f(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExTopStrSeq* deals with a non empty topological space \mathcal{A} and a unary functor \mathcal{F} yielding a subset of \mathcal{A} , and states that:

There exists a sequence S of subsets of the carrier of \mathcal{A} such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

We now state the proposition

(5) Let X be a non empty 1-sorted structure and F be a sequence of subsets of the carrier of X . Then $\text{rng } F$ is a family of subsets of X .

Let X be a non empty 1-sorted structure and let F be a sequence of subsets of the carrier of X . Then $\bigcup F$ is a subset of X . Then $\bigcap F$ is a subset of X .

2. LOWER AND UPPER LIMIT OF SEQUENCES OF SUBSETS

Let X be a non empty set, let S be a function from \mathbb{N} into X , and let k be a natural number. The functor $S \uparrow k$ yields a function from \mathbb{N} into X and is defined as follows:

(Def. 2) For every natural number n holds $(S \uparrow k)(n) = S(n + k)$.

Let X be a non empty 1-sorted structure and let F be a sequence of subsets of the carrier of X . The functor $\liminf F$ yields a subset of X and is defined as follows:

(Def. 3) There exists a sequence f of subsets of X such that $\liminf F = \bigcup f$ and for every natural number n holds $f(n) = \bigcap (F \uparrow n)$.

The functor $\limsup F$ yields a subset of X and is defined by:

(Def. 4) There exists a sequence f of subsets of X such that $\limsup F = \bigcap f$ and for every natural number n holds $f(n) = \bigcup (F \uparrow n)$.

Next we state a number of propositions:

- (6) Let X be a non empty 1-sorted structure, F be a sequence of subsets of the carrier of X , and x be a set. Then $x \in \bigcap F$ if and only if for every natural number z holds $x \in F(z)$.
- (7) Let X be a non empty 1-sorted structure, F be a sequence of subsets of the carrier of X , and x be a set. Then $x \in \liminf F$ if and only if there exists a natural number n such that for every natural number k holds $x \in F(n+k)$.
- (8) Let X be a non empty 1-sorted structure, F be a sequence of subsets of the carrier of X , and x be a set. Then $x \in \limsup F$ if and only if for every natural number n there exists a natural number k such that $x \in F(n+k)$.
- (9) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\liminf F \subseteq \limsup F$.
- (10) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\bigcap F \subseteq \liminf F$.
- (11) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\limsup F \subseteq \bigcup F$.
- (12) For every non empty 1-sorted structure X and for every sequence F of subsets of the carrier of X holds $\liminf F = (\limsup \text{Complement } F)^c$.
- (13) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X . If for every natural number n holds $C(n) = A(n) \cap B(n)$, then $\liminf C = \liminf A \cap \liminf B$.
- (14) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X . If for every natural number n holds $C(n) = A(n) \cup B(n)$, then $\limsup C = \limsup A \cup \limsup B$.
- (15) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X . If for every natural number n holds $C(n) = A(n) \cup B(n)$, then $\liminf A \cup \liminf B \subseteq \liminf C$.
- (16) Let X be a non empty 1-sorted structure and A, B, C be sequences of subsets of the carrier of X . If for every natural number n holds $C(n) = A(n) \cap B(n)$, then $\limsup C \subseteq \limsup A \cap \limsup B$.
- (17) Let X be a non empty 1-sorted structure, A be a sequence of subsets of the carrier of X , and B be a subset of X . If for every natural number n holds $A(n) = B$, then $\limsup A = B$.
- (18) Let X be a non empty 1-sorted structure, A be a sequence of subsets of the carrier of X , and B be a subset of X . If for every natural number n holds $A(n) = B$, then $\liminf A = B$.
- (19) Let X be a non empty 1-sorted structure, A, B be sequences of subsets of the carrier of X , and C be a subset of X . If for every natural number n holds $B(n) = C \dot{-} A(n)$, then $C \dot{-} \liminf A \subseteq \limsup B$.
- (20) Let X be a non empty 1-sorted structure, A, B be sequences of subsets

of the carrier of X , and C be a subset of X . If for every natural number n holds $B(n) = C \dot{-} A(n)$, then $C \dot{-} \limsup A \subseteq \limsup B$.

3. ASCENDING AND DESCENDING FAMILIES OF SUBSETS

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of T . We say that S is descending if and only if:

(Def. 5) For every natural number i holds $S(i+1) \subseteq S(i)$.

We say that S is ascending if and only if:

(Def. 6) For every natural number i holds $S(i) \subseteq S(i+1)$.

Next we state several propositions:

- (21) Let f be a function. Suppose that for every natural number i holds $f(i+1) \subseteq f(i)$. Let i, j be natural numbers. If $i \leq j$, then $f(j) \subseteq f(i)$.
- (22) Let T be a non empty 1-sorted structure and C be a sequence of subsets of T . Suppose C is descending. Let i, m be natural numbers. If $i \geq m$, then $C(i) \subseteq C(m)$.
- (23) Let T be a non empty 1-sorted structure and C be a sequence of subsets of T . Suppose C is ascending. Let i, m be natural numbers. If $i \geq m$, then $C(m) \subseteq C(i)$.
- (24) Let T be a non empty 1-sorted structure, F be a sequence of subsets of T , and x be a set. Suppose F is descending and there exists a natural number k such that for every natural number n such that $n > k$ holds $x \in F(n)$. Then $x \in \bigcap F$.
- (25) Let T be a non empty 1-sorted structure and F be a sequence of subsets of T . If F is descending, then $\liminf F = \bigcap F$.
- (26) Let T be a non empty 1-sorted structure and F be a sequence of subsets of T . If F is ascending, then $\limsup F = \bigcup F$.

4. CONSTANT AND CONVERGENT SEQUENCES

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of T . We say that S is convergent if and only if:

(Def. 7) $\limsup S = \liminf S$.

We now state the proposition

- (27) Let T be a non empty 1-sorted structure and S be a sequence of subsets of T . If S is constant, then the value of S is a subset of T .

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of T . Let us observe that S is constant if and only if:

(Def. 8) There exists a subset A of T such that for every natural number n holds $S(n) = A$.

Let T be a non empty 1-sorted structure. Observe that every sequence of subsets of T which is constant is also convergent, ascending, and descending.

Let T be a non empty 1-sorted structure. Note that there exists a sequence of subsets of T which is constant and non empty.

Let T be a non empty 1-sorted structure and let S be a convergent sequence of subsets of T . The functor $\text{limes } S$ yields a subset of T and is defined as follows:

(Def. 9) $\text{limes } S = \limsup S$ and $\text{limes } S = \liminf S$.

One can prove the following proposition

(28) Let X be a non empty 1-sorted structure, F be a convergent sequence of subsets of X , and x be a set. Then $x \in \text{limes } F$ if and only if there exists a natural number n such that for every natural number k holds $x \in F(n+k)$.

5. TOPOLOGICAL LEMMAS

In the sequel n denotes a natural number.

Let f be a finite sequence of elements of the carrier of \mathcal{E}_T^2 . One can check that $\tilde{\mathcal{L}}(f)$ is closed.

We now state several propositions:

- (29) Let r be a real number, M be a non empty Reflexive metric structure, and x be an element of M . If $0 < r$, then $x \in \text{Ball}(x, r)$.
- (30) For every point x of \mathcal{E}^n and for every real number r holds $\text{Ball}(x, r)$ is an open subset of \mathcal{E}_T^n .
- (31) For all points p, q of \mathcal{E}_T^n and for all points p', q' of \mathcal{E}^n such that $p = p'$ and $q = q'$ holds $\rho(p', q') = |p - q|$.
- (32) Let p be a point of \mathcal{E}^n , x, p' be points of \mathcal{E}_T^n , and r be a real number. If $p = p'$ and $x \in \text{Ball}(p, r)$, then $|x - p'| < r$.
- (33) Let p be a point of \mathcal{E}^n , x, p' be points of \mathcal{E}_T^n , and r be a real number. If $p = p'$ and $|x - p'| < r$, then $x \in \text{Ball}(p, r)$.
- (34) Let n be a natural number, r be a point of \mathcal{E}_T^n , and X be a subset of \mathcal{E}_T^n . Suppose $r \in \overline{X}$. Then there exists a sequence s_1 in \mathcal{E}_T^n such that $\text{rng } s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 = r$.

Let M be a non empty metric space. Note that M_{top} is first-countable.

Let n be a natural number. Note that \mathcal{E}_T^n is first-countable.

Next we state several propositions:

- (35) Let p be a point of \mathcal{E}^n , q be a point of \mathcal{E}_T^n , and r be a real number. If $p = q$ and $r > 0$, then $\text{Ball}(p, r)$ is a neighbourhood of q .

- (36) Let A be a subset of \mathcal{E}_T^n , p be a point of \mathcal{E}_T^n , and p' be a point of \mathcal{E}^n . Suppose $p = p'$. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ holds $\text{Ball}(p', r)$ meets A .
- (37) Let x, y be points of \mathcal{E}_T^n and x' be a point of \mathcal{E}^n . If $x' = x$ and $x \neq y$, then there exists a real number r such that $y \notin \text{Ball}(x', r)$.
- (38) Let S be a subset of \mathcal{E}_T^n . Then S is non Bounded if and only if for every real number r such that $r > 0$ there exist points x, y of \mathcal{E}^n such that $x \in S$ and $y \in S$ and $\rho(x, y) > r$.
- (39) For all real numbers a, b and for all points x, y of \mathcal{E}^n such that $\text{Ball}(x, a)$ meets $\text{Ball}(y, b)$ holds $\rho(x, y) < a + b$.
- (40) Let a, b, c be real numbers and x, y, z be points of \mathcal{E}^n . If $\text{Ball}(x, a)$ meets $\text{Ball}(z, c)$ and $\text{Ball}(z, c)$ meets $\text{Ball}(y, b)$, then $\rho(x, y) < a + b + 2 \cdot c$.
- (41) Let X, Y be non empty topological spaces, x be a point of X , y be a point of Y , and V be a subset of $\{X, Y\}$. Then V is a neighbourhood of $\{\{x\}, \{y\}\}$ if and only if V is a neighbourhood of $\langle x, y \rangle$.

Now we present two schemes. The scheme *TSubsetEx* deals with a non empty topological structure \mathcal{A} and a unary predicate \mathcal{P} , and states that:

There exists a subset X of \mathcal{A} such that for every point x of \mathcal{A} holds $x \in X$ iff $\mathcal{P}[x]$

for all values of the parameters.

The scheme *TSubsetUniq* deals with a topological structure \mathcal{A} and a unary predicate \mathcal{P} , and states that:

Let A_1, A_2 be subsets of \mathcal{A} . Suppose for every point x of \mathcal{A} holds $x \in A_1$ iff $\mathcal{P}[x]$ and for every point x of \mathcal{A} holds $x \in A_2$ iff $\mathcal{P}[x]$.

Then $A_1 = A_2$

for all values of the parameters.

Let T be a non empty topological structure, let S be a sequence of subsets of the carrier of T , and let i be a natural number. Then $S(i)$ is a subset of T .

One can prove the following two propositions:

- (42) Let T be a non empty 1-sorted structure, S be a sequence of subsets of the carrier of T , and R be a sequence of naturals. Then $S \cdot R$ is a sequence of subsets of T .
- (43) $\text{id}_{\mathbb{N}}$ is an increasing sequence of naturals.

Let us observe that $\text{id}_{\mathbb{N}}$ is real-yielding.

6. SUBSEQUENCES

Let T be a non empty 1-sorted structure and let S be a sequence of subsets of the carrier of T . A sequence of subsets of T is said to be a subsequence of S if:

(Def. 10) There exists an increasing sequence N_1 of naturals such that it = $S \cdot N_1$.

We now state several propositions:

- (44) For every non empty 1-sorted structure T holds every sequence S of subsets of the carrier of T is a subsequence of S .
- (45) Let T be a non empty 1-sorted structure, S be a sequence of subsets of T , and S_1 be a subsequence of S . Then $\text{rng } S_1 \subseteq \text{rng } S$.
- (46) Let T be a non empty 1-sorted structure, S_1 be a sequence of subsets of the carrier of T , and S_2 be a subsequence of S_1 . Then every subsequence of S_2 is a subsequence of S_1 .
- (47) Let T be a non empty 1-sorted structure, F, G be sequences of subsets of the carrier of T , and A be a subset of T . Suppose G is a subsequence of F and for every natural number i holds $F(i) = A$. Then $G = F$.
- (48) Let T be a non empty 1-sorted structure, A be a constant sequence of subsets of T , and B be a subsequence of A . Then $A = B$.
- (49) Let T be a non empty 1-sorted structure, S be a sequence of subsets of the carrier of T , R be a subsequence of S , and n be a natural number. Then there exists a natural number m such that $m \geq n$ and $R(n) = S(m)$.

Let T be a non empty 1-sorted structure and let X be a constant sequence of subsets of T . Note that every subsequence of X is constant.

The scheme *SubSeqChoice* deals with a non empty topological space \mathcal{A} , a sequence \mathcal{B} of subsets of the carrier of \mathcal{A} , and a unary predicate \mathcal{P} , and states that:

There exists a subsequence S_1 of \mathcal{B} such that for every natural number n holds $\mathcal{P}[S_1(n)]$

provided the following condition is satisfied:

- For every natural number n there exists a natural number m such that $n \leq m$ and $\mathcal{P}[\mathcal{B}(m)]$.

7. THE LOWER TOPOLOGICAL LIMIT

Let T be a non empty topological space and let S be a sequence of subsets of the carrier of T . The functor $\text{Li } S$ yielding a subset of T is defined by the condition (Def. 11).

(Def. 11) Let p be a point of T . Then $p \in \text{Li } S$ if and only if for every neighbourhood G of p there exists a natural number k such that for every natural number m such that $m > k$ holds $S(m)$ meets G .

The following propositions are true:

- (50) Let S be a sequence of subsets of the carrier of \mathcal{E}_T^n , p be a point of \mathcal{E}_T^n , and p' be a point of \mathcal{E}^n . Suppose $p = p'$. Then $p \in \text{Li } S$ if and only if for every real number r such that $r > 0$ there exists a natural number k

such that for every natural number m such that $m > k$ holds $S(m)$ meets $\text{Ball}(p', r)$.

- (51) For every non empty topological space T and for every sequence S of subsets of the carrier of T holds $\overline{\text{Li } S} = \text{Li } S$.
- (52) For every non empty topological space T and for every sequence S of subsets of the carrier of T holds $\text{Li } S$ is closed.
- (53) Let T be a non empty topological space and R, S be sequences of subsets of the carrier of T . If R is a subsequence of S , then $\text{Li } S \subseteq \text{Li } R$.
- (54) Let T be a non empty topological space and A, B be sequences of subsets of the carrier of T . If for every natural number i holds $A(i) \subseteq B(i)$, then $\text{Li } A \subseteq \text{Li } B$.
- (55) Let T be a non empty topological space and A, B, C be sequences of subsets of the carrier of T . If for every natural number i holds $C(i) = A(i) \cup B(i)$, then $\text{Li } A \cup \text{Li } B \subseteq \text{Li } C$.
- (56) Let T be a non empty topological space and A, B, C be sequences of subsets of the carrier of T . If for every natural number i holds $C(i) = A(i) \cap B(i)$, then $\text{Li } C \subseteq \text{Li } A \cap \text{Li } B$.
- (57) Let T be a non empty topological space and F, G be sequences of subsets of the carrier of T . If for every natural number i holds $G(i) = \overline{F(i)}$, then $\text{Li } G = \text{Li } F$.
- (58) Let S be a sequence of subsets of the carrier of \mathcal{E}_T^n and p be a point of \mathcal{E}_T^n . Given a sequence s in \mathcal{E}_T^n such that s is convergent and for every natural number x holds $s(x) \in S(x)$ and $p = \lim s$. Then $p \in \text{Li } S$.
- (59) Let T be a non empty topological space, P be a subset of T , and s be a sequence of subsets of the carrier of T . If for every natural number i holds $s(i) \subseteq P$, then $\text{Li } s \subseteq \overline{P}$.
- (60) Let T be a non empty topological space, F be a sequence of subsets of the carrier of T , and A be a subset of T . If for every natural number i holds $F(i) = A$, then $\text{Li } F = \overline{A}$.
- (61) Let T be a non empty topological space, F be a sequence of subsets of the carrier of T , and A be a closed subset of T . If for every natural number i holds $F(i) = A$, then $\text{Li } F = A$.
- (62) Let S be a sequence of subsets of the carrier of \mathcal{E}_T^n and P be a subset of \mathcal{E}_T^n . Suppose P is Bounded and for every natural number i holds $S(i) \subseteq P$. Then $\text{Li } S$ is Bounded.
- (63) Let S be a sequence of subsets of the carrier of \mathcal{E}_T^2 and P be a subset of \mathcal{E}_T^2 . Suppose P is Bounded and for every natural number i holds $S(i) \subseteq P$ and for every natural number i holds $S(i)$ is compact. Then $\text{Li } S$ is compact.
- (64) Let A, B be sequences of subsets of the carrier of \mathcal{E}_T^n and C be a sequence of subsets of the carrier of $\{\mathcal{E}_T^n, \mathcal{E}_T^n\}$. If for every natural number i holds

- $C(i) = [A(i), B(i)]$, then $[Li A, Li B] = Li C$.
- (65) For every sequence S of subsets of \mathcal{E}_T^2 holds $\liminf S \subseteq Li S$.
 - (66) For every simple closed curve C and for every natural number i holds $Fr((UBD \tilde{\mathcal{L}}(Cage(C, i)))^c) = \tilde{\mathcal{L}}(Cage(C, i))$.

8. THE UPPER TOPOLOGICAL LIMIT

Let T be a non empty topological space and let S be a sequence of subsets of the carrier of T . The functor $Ls S$ yields a subset of T and is defined as follows:

(Def. 12) For every set x holds $x \in Ls S$ iff there exists a subsequence A of S such that $x \in Li A$.

One can prove the following propositions:

- (67) Let N be a natural number, F be a sequence of \mathcal{E}_T^N , x be a point of \mathcal{E}_T^N , and x' be a point of \mathcal{E}^N . Suppose $x = x'$. Then x is a cluster point of F if and only if for every real number r and for every natural number n such that $r > 0$ there exists a natural number m such that $n \leq m$ and $F(m) \in Ball(x', r)$.
- (68) For every non empty topological space T and for every sequence A of subsets of the carrier of T holds $Li A \subseteq Ls A$.
- (69) Let A, B, C be sequences of subsets of the carrier of \mathcal{E}_T^2 . Suppose for every natural number i holds $A(i) \subseteq B(i)$ and C is a subsequence of A . Then there exists a subsequence D of B such that for every natural number i holds $C(i) \subseteq D(i)$.
- (70) Let A, B, C be sequences of subsets of the carrier of \mathcal{E}_T^2 . Suppose for every natural number i holds $A(i) \subseteq B(i)$ and C is a subsequence of B . Then there exists a subsequence D of A such that for every natural number i holds $D(i) \subseteq C(i)$.
- (71) Let A, B be sequences of subsets of the carrier of \mathcal{E}_T^2 . If for every natural number i holds $A(i) \subseteq B(i)$, then $Ls A \subseteq Ls B$.
- (72) Let A, B, C be sequences of subsets of the carrier of \mathcal{E}_T^2 . If for every natural number i holds $C(i) = A(i) \cup B(i)$, then $Ls A \cup Ls B \subseteq Ls C$.
- (73) Let A, B, C be sequences of subsets of the carrier of \mathcal{E}_T^2 . If for every natural number i holds $C(i) = A(i) \cap B(i)$, then $Ls C \subseteq Ls A \cap Ls B$.
- (74) Let A, B be sequences of subsets of the carrier of \mathcal{E}_T^2 and C, C_1 be sequences of subsets of the carrier of $[\mathcal{E}_T^2, \mathcal{E}_T^2]$. Suppose for every natural number i holds $C(i) = [A(i), B(i)]$ and C_1 is a subsequence of C . Then there exist sequences A_1, B_1 of subsets of the carrier of \mathcal{E}_T^2 such that A_1 is a subsequence of A and B_1 is a subsequence of B and for every natural number i holds $C_1(i) = [A_1(i), B_1(i)]$.

- (75) Let A, B be sequences of subsets of the carrier of \mathcal{E}_T^2 and C be a sequence of subsets of the carrier of $\{ \mathcal{E}_T^2, \mathcal{E}_T^2 \}$. If for every natural number i holds $C(i) = \{ A(i), B(i) \}$, then $\text{Ls } C \subseteq \{ \text{Ls } A, \text{Ls } B \}$.
- (76) Let T be a non empty topological space, F be a sequence of subsets of the carrier of T , and A be a subset of T . If for every natural number i holds $F(i) = A$, then $\text{Li } F = \text{Ls } F$.
- (77) Let F be a sequence of subsets of the carrier of \mathcal{E}_T^2 and A be a subset of \mathcal{E}_T^2 . If for every natural number i holds $F(i) = A$, then $\text{Ls } F = \overline{A}$.
- (78) Let F, G be sequences of subsets of the carrier of \mathcal{E}_T^2 . If for every natural number i holds $G(i) = \overline{F(i)}$, then $\text{Ls } G = \text{Ls } F$.

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