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Lattice of Fuzzy Sets[1]

Takashi Mitsuishi
Miyagi University

Grzegorz Bancerek
Białystok Technical University

Summary. This article concerns a connection of fuzzy logic and lattice theory. Namely, the fuzzy sets form a Heyting lattice with union and intersection of fuzzy sets as meet and join operations. The lattice of fuzzy sets is defined as the product of interval posets. As the final result, we have characterized the composition of fuzzy relations in terms of lattice theory and proved its associativity.

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The notation and terminology used in this paper are introduced in the following articles: [18], [9], [23], [6], [7], [17], [1], [8], [22], [16], [20], [15], [24], [21], [14], [19], [2], [3], [4], [12], [10], [5], [13], and [11].

1. Posets of Real Numbers

Let R be a relational structure. We say that R is real if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) The carrier of $R \subseteq \mathbb{R}$, and

(ii) for all real numbers x, y such that $x \in$ the carrier of R and $y \in$ the carrier of R holds $\langle x, y\rangle \in$ the internal relation of R iff $x \leqslant y$.

Let R be a relational structure. We say that R is interval if and only if:

(Def. 2) R is real and there exist real numbers a, b such that $a \leqslant b$ and the carrier of $R = [a, b]$.

Let us mention that every relational structure which is interval is also real and non empty.

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Let us observe that every relational structure which is empty is also real.

One can prove the following proposition

(1) For every subset X of $\mathbb{R}$ there exists a strict relational structure R such that the carrier of $R = X$ and R is real.

Let us note that there exists a relational structure which is interval and strict.

The following proposition is true

(2) Let R_1, R_2 be real relational structures. Suppose the carrier of $R_1 =$ the carrier of R_2. Then the relational structure of $R_1 =$ the relational structure of R_2.

Let R be a non empty real relational structure. Observe that every element of R is real.

Let X be a subset of $\mathbb{R}$. The functor RealPoset X yields a real strict relational structure and is defined as follows:

(Def. 3) The carrier of RealPoset $X = X$.

Let X be a non empty subset of $\mathbb{R}$. Note that RealPoset X is non empty.

Let R be a relational structure and let x, y be elements of R. We introduce $x \preceq y$ and $y \succeq x$ as synonyms of $x \leqslant y$.

Let x, y be real numbers. We introduce $x \leqslant_{\mathbb{R}} y$ and $y \geqslant_{\mathbb{R}} x$ as synonyms of $x \leqslant y$. We introduce $y <_{\mathbb{R}} x$ and $x >_{\mathbb{R}} y$ as antonyms of $x \leqslant y$.

We now state the proposition

(3) For every non empty real relational structure R and for all elements x, y of R holds $x \leqslant_{\mathbb{R}} y$ iff $x \preceq y$.

Let us observe that every relational structure which is real is also reflexive, antisymmetric, and transitive.

Let us observe that every real non empty relational structure is connected.

Let R be a non empty real relational structure and let x, y be elements of R. Then $\max(x, y)$ is an element of R.

Let R be a non empty real relational structure and let x, y be elements of R. Then $\min(x, y)$ is an element of R.

Let us note that every real non empty relational structure has l.u.b.'s and g.l.b.'s.

We follow the rules: x, y denote real numbers, R denotes a real non empty relational structure, and a, b denote elements of R.

One can prove the following four propositions:

(4) $a \sqcup b = \max(a, b)$.

(5) $a \sqcap b = \min(a, b)$.

(6) There exists x such that $x \in$ the carrier of R and for every y such that $y \in$ the carrier of R holds $x \leqslant y$ if and only if R is lower-bounded.

(7) There exists x such that $x \in$ the carrier of R and for every y such that $y \in$ the carrier of R holds $x \geqslant y$ if and only if R is upper-bounded.

Let us observe that every non empty relational structure which is interval is also bounded.

The following proposition is true

(8) For every interval non empty relational structure R and for every set X holds sup X exists in R.

Let us observe that every interval non empty relational structure is complete.

Let us note that every chain is distributive.

One can check that every interval non empty relational structure is Heyting.

One can verify that $[0, 1]$ is non empty.

Let us observe that RealPoset$[0, 1]$ is interval.

2. Product of Heyting Lattices

We now state several propositions:

(9) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a sup-semilattice. Then $\prod J$ has l.u.b.'s.

(10) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a semilattice. Then $\prod J$ has g.l.b.'s.

(11) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a semilattice. Let f, g be elements of $\prod J$ and i be an element of I. Then $(f \sqcap g)(i) = f(i) \sqcap g(i)$.

(12) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a sup-semilattice. Let f, g be elements of $\prod J$ and i be an element of I. Then $(f \sqcup g)(i) = f(i) \sqcup g(i)$.

(13) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a Heyting complete lattice. Then $\prod J$ is complete and Heyting.

Let A be a non empty set and let R be a complete Heyting lattice. Observe that R^A is Heyting.

3. Lattice of Fuzzy Sets

Let A be a non empty set. The functor FuzzyLattice A yielding a Heyting complete lattice is defined by:

(Def. 4) $\text{FuzzyLattice}\,A = (\text{RealPoset}[0,1])^A$.

We now state the proposition

(14) For every non empty set A holds the carrier of $\text{FuzzyLattice}\,A = [0,1]^A$.

Let A be a non empty set. Note that FuzzyLattice A is constituted functions.

Next we state the proposition

(15) Let R be a complete Heyting lattice, X be a subset of R, and y be an element of R. Then $\bigsqcup_R X \sqcap y = \bigsqcup_R \{x \sqcap y; x$ ranges over elements of R: $x \in X\}$.

Let X be a non empty set and let a be an element of FuzzyLattice X. The functor ${}^{@}a$ yields a membership function of X and is defined by:

(Def. 5) ${}^{@}a = a$.

Let X be a non empty set and let f be a membership function of X. The functor $f^{@}$ yielding an element of FuzzyLattice X is defined by:

(Def. 6) $f^{@} = f$.

Let X be a non empty set, let f be a membership function of X, and let x be an element of X. Then $f(x)$ is an element of $\text{RealPoset}[0,1]$.

Let X be a non empty set, let f be an element of FuzzyLattice X, and let x be an element of X. Then $f(x)$ is an element of $\text{RealPoset}[0,1]$.

For simplicity, we follow the rules: C is a non empty set, c is an element of C, f, g are membership functions of C, and s, t are elements of FuzzyLattice C.

Next we state several propositions:

(16) For every c holds $f(c) \leqslant_{\mathbb{R}} g(c)$ iff $f^{@} \preceq g^{@}$.

(17) $s \preceq t$ iff for every c holds $({}^{@}s)(c) \leqslant_{\mathbb{R}} ({}^{@}t)(c)$.

(18) $\max(f, g) = f^{@} \sqcup g^{@}$.

(19) $s \sqcup t = \max({}^{@}s, {}^{@}t)$.

(20) $\min(f, g) = f^{@} \sqcap g^{@}$.

(21) $s \sqcap t = \min({}^{@}s, {}^{@}t)$.

4. Associativity of Composition of Fuzzy Relations

In this article we present several logical schemes. The scheme *SupDistributivity* deals with a complete lattice $\mathcal{A}$, non empty sets $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); y$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y]\}; x$ ranges over elements of $\mathcal{B} : \mathcal{P}[x]\} = \bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); x$ ranges over elements of $\mathcal{B}, y$ ranges over elements of $\mathcal{C} : \mathcal{P}[x] \wedge \mathcal{Q}[y]\}$

for all values of the parameters.

The scheme *SupDistributivity'* deals with a complete lattice $\mathcal{A}$, non empty sets $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); x$ ranges over elements of $\mathcal{B} : \mathcal{P}[x]\}; y$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y]\} = \bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); x$ ranges over elements of $\mathcal{B}, y$ ranges over elements of $\mathcal{C} : \mathcal{P}[x] \wedge \mathcal{Q}[y]\}$

for all values of the parameters.

The scheme *FraenkelF'R'* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, two binary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and a binary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(u_1,v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1,v_1]\} = \{\mathcal{G}(u_2,v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_2,v_2]\}$

provided the parameters meet the following condition:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u,v]$ holds $\mathcal{F}(u,v) = \mathcal{G}(u,v)$.

The scheme *FraenkelF6"R* deals with a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, two binary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{\mathcal{F}(u_1,v_1); u_1$ ranges over elements of $\mathcal{A}, v_1$ ranges over elements of $\mathcal{B} : \mathcal{P}[u_1,v_1]\} = \{\mathcal{G}(u_2,v_2); u_2$ ranges over elements of $\mathcal{A}, v_2$ ranges over elements of $\mathcal{B} : \mathcal{Q}[u_2,v_2]\}$

provided the following requirements are met:

- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ holds $\mathcal{P}[u,v]$ iff $\mathcal{Q}[u,v]$, and
- For every element u of $\mathcal{A}$ and for every element v of $\mathcal{B}$ such that $\mathcal{P}[u,v]$ holds $\mathcal{F}(u,v) = \mathcal{G}(u,v)$.

The scheme *SupCommutativity* deals with a complete lattice $\mathcal{A}$, non empty sets $\mathcal{B}$, $\mathcal{C}$, two binary functors $\mathcal{F}$ and $\mathcal{G}$ yielding elements of $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(x,y); y$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y]\}; x$ ranges over elements of $\mathcal{B} : \mathcal{P}[x]\} = \bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{G}(x',y'); x'$ ranges over elements of $\mathcal{B} : \mathcal{P}[x']\}; y'$ ranges over elements of $\mathcal{C} : \mathcal{Q}[y']\}$

provided the parameters meet the following condition:

- For every element x of $\mathcal{B}$ and for every element y of $\mathcal{C}$ such that $\mathcal{P}[x]$ and $\mathcal{Q}[y]$ holds $\mathcal{F}(x,y) = \mathcal{G}(x,y)$.

One can prove the following propositions:

(22) Let X, Y, Z be non empty sets, R be a membership function of X, Y, S be a membership function of Y, Z, x be an element of X, and z be an element of Z. Then $(R\,S)(\langle x, z\rangle) = \bigsqcup_{\mathrm{RealPoset}[0,1]}\{R(\langle x, y\rangle) \sqcap S(\langle y, z\rangle) : y$ ranges over elements of $Y\}$.

(23) Let X, Y, Z, W be non empty sets, R be a membership function of X, Y, S be a membership function of Y, Z, and T be a membership function of Z, W. Then $(R\,S)\,T = R\,(S\,T)$.

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