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Banach Space of Absolute Summable Real Sequences

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Summary. A continuation of [5]. As the example of real norm spaces, we introduce the arithmetic addition and multiplication in the set of absolute summable real sequences and also introduce the norm. This set has the structure of the Banach space.

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The notation and terminology used here are introduced in the following papers: [14], [17], [4], [1], [13], [7], [2], [3], [18], [16], [10], [15], [11], [9], [8], [12], and [6].

1. The Space of Absolute Summable Real Sequences

The subset the set of l1-real sequences of the linear space of real sequences is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ the set of l1-real sequences if and only if $x \in$ the set of real sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is absolutely summable.

Let us observe that the set of l1-real sequences is non empty.

One can prove the following two propositions:

(1) The set of l1-real sequences is linearly closed.

(2) $\langle$the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space

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of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences)⟩ is a subspace of the linear space of real sequences.

One can check that ⟨the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences)⟩ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

One can prove the following proposition

(3) ⟨the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences)⟩ is a real linear space.

The function $\mathrm{norm}_{\mathrm{seq}}$ from the set of l1-real sequences into $\mathbb{R}$ is defined by:

(Def. 2) For every set x such that $x \in$ the set of l1-real sequences holds $\mathrm{norm}_{\mathrm{seq}}(x) = \sum|\mathrm{id}_{\mathrm{seq}}(x)|$.

Let X be a non empty set, let Z be an element of X, let A be a binary operation on X, let M be a function from $[\!:\mathbb{R}, X:\!]$ into X, and let N be a function from X into $\mathbb{R}$. One can check that $\langle X, Z, A, M, N \rangle$ is non empty.

Next we state four propositions:

(4) Let l be a normed structure. Suppose ⟨the carrier of l, the zero of l, the addition of l, the external multiplication of l⟩ is a real linear space. Then l is a real linear space.

(5) Let r_1 be a sequence of real numbers. Suppose that for every natural number n holds $r_1(n) = 0$. Then r_1 is absolutely summable and $\sum|r_1| = 0$.

(6) Let r_1 be a sequence of real numbers. Suppose r_1 is absolutely summable and $\sum|r_1| = 0$. Let n be a natural number. Then $r_1(n) = 0$.

(7) ⟨the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences), $\mathrm{norm}_{\mathrm{seq}}$⟩ is a real linear space.

The non empty normed structure l1-Space is defined by the condition (Def. 3).

(Def. 3) l1-Space = ⟨the set of l1-real sequences, Zero_(the set of l1-real sequences, the linear space of real sequences), Add_(the set of l1-real sequences, the linear space of real sequences), Mult_(the set of l1-real sequences, the linear space of real sequences), $\mathrm{norm}_{\mathrm{seq}}$⟩.

2. The Space is Banach Space

One can prove the following two propositions:

(8) The carrier of l1-Space = the set of l1-real sequences and for every set x holds x is an element of l1-Space iff x is a sequence of real numbers and $\mathrm{id}_{\mathrm{seq}}(x)$ is absolutely summable and for every set x holds x is a vector of l1-Space iff x is a sequence of real numbers and $\mathrm{id}_{\mathrm{seq}}(x)$ is absolutely summable and $0_{\text{l1-Space}} = \text{Zeroseq}$ and for every vector u of l1-Space holds $u = \mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of l1-Space holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$ and for every real number r and for every vector u of l1-Space holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$ and for every vector u of l1-Space holds $-u = -\mathrm{id}_{\mathrm{seq}}(u)$ and $\mathrm{id}_{\mathrm{seq}}(-u) = -\mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of l1-Space holds $u - v = \mathrm{id}_{\mathrm{seq}}(u) - \mathrm{id}_{\mathrm{seq}}(v)$ and for every vector v of l1-Space holds $\mathrm{id}_{\mathrm{seq}}(v)$ is absolutely summable and for every vector v of l1-Space holds $\|v\| = \sum |\mathrm{id}_{\mathrm{seq}}(v)|$.

(9) Let x, y be points of l1-Space and a be a real number. Then $\|x\| = 0$ iff $x = 0_{\text{l1-Space}}$ and $0 \leqslant \|x\|$ and $\|x + y\| \leqslant \|x\| + \|y\|$ and $\|a \cdot x\| = |a| \cdot \|x\|$.

Let us observe that l1-Space is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

Let X be a non empty normed structure and let x, y be points of X. The functor $\rho(x, y)$ yields a real number and is defined by:

(Def. 4) $\rho(x, y) = \|x - y\|$.

Let N_1 be a non empty normed structure and let s_1 be a sequence of N_1. We say that s_1 is CCauchy if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let r_2 be a real number. Suppose $r_2 > 0$. Then there exists a natural number k_1 such that for all natural numbers n_1, m_1 if $n_1 \geqslant k_1$ and $m_1 \geqslant k_1$, then $\rho(s_1(n_1), s_1(m_1)) < r_2$.

We introduce s_1 is Cauchy sequence by norm as a synonym of s_1 is CCauchy.

In the sequel N_1 denotes a non empty real normed space and s_2 denotes a sequence of N_1.

We now state two propositions:

(10) s_2 is Cauchy sequence by norm if and only if for every real number r such that $r > 0$ there exists a natural number k such that for all natural numbers n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|s_2(n) - s_2(m)\| < r$.

(11) For every sequence v_1 of l1-Space such that v_1 is Cauchy sequence by norm holds v_1 is convergent.

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