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Banach Space of Bounded Real Sequences

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Summary. We introduce the arithmetic addition and multiplication in the set of bounded real sequences and also introduce the norm. This set has the structure of the Banach space.

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The articles [23], [6], [27], [29], [28], [15], [21], [3], [1], [2], [20], [24], [9], [4], [5], [7], [26], [22], [16], [17], [14], [11], [12], [10], [25], [13], [8], [19], and [18] provide the notation and terminology for this paper.

1. The Banach Space of Bounded Real Sequences

The subset the set of bounded real sequences of the linear space of real sequences is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ the set of bounded real sequences if and only if $x \in$ the set of real sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is bounded.

Let us note that the set of bounded real sequences is non empty and the set of bounded real sequences is linearly closed.

One can prove the following proposition

(1) $\langle$the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences)$\rangle$ is a subspace of the linear space of real sequences.

One can verify that $\langle$the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded

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real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences)⟩ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

The function linfty-norm from the set of bounded real sequences into $\mathbb{R}$ is defined by:

(Def. 2) For every set x such that $x \in$ the set of bounded real sequences holds linfty-norm$(x) = \sup \mathrm{rng}|\mathrm{id}_{\mathrm{seq}}(x)|$.

The following proposition is true

(2) Let r_1 be a sequence of real numbers. Then r_1 is bounded and $\sup \mathrm{rng}|r_1| = 0$ if and only if for every natural number n holds $r_1(n) = 0$.

Let us mention that ⟨the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences), linfty-norm⟩ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

The non empty normed structure linfty-Space is defined by the condition (Def. 3).

(Def. 3) linfty-Space = ⟨the set of bounded real sequences, Zero_(the set of bounded real sequences, the linear space of real sequences), Add_(the set of bounded real sequences, the linear space of real sequences), Mult_(the set of bounded real sequences, the linear space of real sequences), linfty-norm⟩.

We now state two propositions:

(3) The carrier of linfty-Space = the set of bounded real sequences and for every set x holds x is a vector of linfty-Space iff x is a sequence of real numbers and $\mathrm{id}_{\mathrm{seq}}(x)$ is bounded and $0_{\text{linfty-Space}} =$ Zeroseq and for every vector u of linfty-Space holds $u = \mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of linfty-Space holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$ and for every real number r and for every vector u of linfty-Space holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$ and for every vector u of linfty-Space holds $-u = -\mathrm{id}_{\mathrm{seq}}(u)$ and $\mathrm{id}_{\mathrm{seq}}(-u) = -\mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of linfty-Space holds $u - v = \mathrm{id}_{\mathrm{seq}}(u) - \mathrm{id}_{\mathrm{seq}}(v)$ and for every vector v of linfty-Space holds $\mathrm{id}_{\mathrm{seq}}(v)$ is bounded and for every vector v of linfty-Space holds $\|v\| = \sup \mathrm{rng}|\mathrm{id}_{\mathrm{seq}}(v)|$.

(4) Let x, y be points of linfty-Space and a be a real number. Then $\|x\| = 0$ iff $x = 0_{\text{linfty-Space}}$ and $0 \leqslant \|x\|$ and $\|x+y\| \leqslant \|x\|+\|y\|$ and $\|a \cdot x\| = |a| \cdot \|x\|$.

Let us observe that linfty-Space is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

Next we state the proposition

(5) For every sequence v_1 of linfty-Space such that v_1 is Cauchy sequence by norm holds v_1 is convergent.

2. The Banach Space of Bounded Functions

Let X be a non empty set, let Y be a real normed space, and let I_1 be a function from X into the carrier of Y. We say that I_1 is bounded if and only if:

(Def. 4) There exists a real number K such that $0 \leqslant K$ and for every element x of X holds $\|I_1(x)\| \leqslant K$.

The following proposition is true

(6) Let X be a non empty set, Y be a real normed space, and f be a function from X into the carrier of Y. If for every element x of X holds $f(x) = 0_Y$, then f is bounded.

Let X be a non empty set and let Y be a real normed space. Note that there exists a function from X into the carrier of Y which is bounded.

Let X be a non empty set and let Y be a real normed space. The functor $\mathrm{BdFuncs}(X, Y)$ yields a subset of $\mathrm{RealVectSpace}(X, Y)$ and is defined by:

(Def. 5) For every set x holds $x \in \mathrm{BdFuncs}(X, Y)$ iff x is a bounded function from X into the carrier of Y.

Let X be a non empty set and let Y be a real normed space. Observe that $\mathrm{BdFuncs}(X, Y)$ is non empty.

The following propositions are true:

(7) For every non empty set X and for every real normed space Y holds $\mathrm{BdFuncs}(X, Y)$ is linearly closed.

(8) For every non empty set X and for every real normed space Y holds $\langle \mathrm{BdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)),$ $\mathrm{Add\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{BdFuncs}(X, Y),$ $\mathrm{RealVectSpace}(X, Y))\rangle$ is a subspace of $\mathrm{RealVectSpace}(X, Y)$.

Let X be a non empty set and let Y be a real normed space. One can verify that $\langle \mathrm{BdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)),$ $\mathrm{Add\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{BdFuncs}(X, Y),$ $\mathrm{RealVectSpace}(X, Y))\rangle$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

One can prove the following proposition

(9) For every non empty set X and for every real normed space Y holds $\langle \mathrm{BdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)),$ $\mathrm{Add\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{BdFuncs}(X, Y),$ $\mathrm{RealVectSpace}(X, Y))\rangle$ is a real linear space.

Let X be a non empty set and let Y be a real normed space. The set of bounded real sequences from X into Y yields a real linear space and is defined as follows:

(Def. 6) The set of bounded real sequences from X into $Y = \langle \mathrm{BdFuncs}(X, Y),$ $\mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Add\_}(\mathrm{BdFuncs}(X, Y),$

RealVectSpace(X, Y)), Mult_(BdFuncs(X, Y), RealVectSpace(X, Y))⟩.

Let X be a non empty set and let Y be a real normed space. Observe that the set of bounded real sequences from X into Y is strict.

One can prove the following three propositions:

(10) Let X be a non empty set, Y be a real normed space, f, g, h be vectors of the set of bounded real sequences from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(11) Let X be a non empty set, Y be a real normed space, f, h be vectors of the set of bounded real sequences from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $h' = h$. Let a be a real number. Then $h = a \cdot f$ if and only if for every element x of X holds $h'(x) = a \cdot f'(x)$.

(12) Let X be a non empty set and Y be a real normed space. Then $0_{\text{the set of bounded real sequences from } X \text{ into } Y} = X \longmapsto 0_Y$.

Let X be a non empty set, let Y be a real normed space, and let f be a set. Let us assume that $f \in$ BdFuncs(X, Y). The functor modetrans(f, X, Y) yields a bounded function from X into the carrier of Y and is defined as follows:

(Def. 7) modetrans$(f, X, Y) = f$.

Let X be a non empty set, let Y be a real normed space, and let u be a function from X into the carrier of Y. The functor PreNorms(u) yielding a non empty subset of $\mathbb{R}$ is defined as follows:

(Def. 8) PreNorms$(u) = \{\|u(t)\| : t$ ranges over elements of $X\}$.

Next we state three propositions:

(13) Let X be a non empty set, Y be a real normed space, and g be a bounded function from X into the carrier of Y. Then PreNorms(g) is non empty and upper bounded.

(14) Let X be a non empty set, Y be a real normed space, and g be a function from X into the carrier of Y. Then g is bounded if and only if PreNorms(g) is upper bounded.

(15) Let X be a non empty set and Y be a real normed space. Then there exists a function N_1 from BdFuncs(X, Y) into $\mathbb{R}$ such that for every set f if $f \in$ BdFuncs(X, Y), then $N_1(f) = \sup$ PreNorms(modetrans(f, X, Y)).

Let X be a non empty set and let Y be a real normed space. The functor BdFuncsNorm(X, Y) yielding a function from BdFuncs(X, Y) into $\mathbb{R}$ is defined by:

(Def. 9) For every set x such that $x \in$ BdFuncs(X, Y) holds BdFuncsNorm$(X, Y)(x) = \sup$ PreNorms(modetrans(x, X, Y)).

One can prove the following two propositions:

(16) Let X be a non empty set, Y be a real normed space, and f be a bounded function from X into the carrier of Y. Then $\mathrm{modetrans}(f, X, Y) = f$.

(17) Let X be a non empty set, Y be a real normed space, and f be a bounded function from X into the carrier of Y. Then $\mathrm{BdFuncsNorm}(X, Y)(f) = \sup \mathrm{PreNorms}(f)$.

Let X be a non empty set and let Y be a real normed space. The real normed space of bounded functions from X into Y yielding a non empty normed structure is defined as follows:

(Def. 10) The real normed space of bounded functions from X into Y $= \langle \mathrm{BdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Add\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{BdFuncs}(X, Y), \mathrm{RealVectSpace}(X, Y)), \mathrm{BdFuncsNorm}(X, Y) \rangle$.

We now state several propositions:

(18) Let X be a non empty set and Y be a real normed space. Then $X \longmapsto 0_Y = 0_{\text{the real normed space of bounded functions from } X \text{ into } Y}$.

(19) Let X be a non empty set, Y be a real normed space, f be a point of the real normed space of bounded functions from X into Y, and g be a bounded function from X into the carrier of Y. If $g = f$, then for every element t of X holds $\|g(t)\| \leqslant \|f\|$.

(20) Let X be a non empty set, Y be a real normed space, and f be a point of the real normed space of bounded functions from X into Y. Then $0 \leqslant \|f\|$.

(21) Let X be a non empty set, Y be a real normed space, and f be a point of the real normed space of bounded functions from X into Y. Suppose $f = 0_{\text{the real normed space of bounded functions from } X \text{ into } Y}$. Then $0 = \|f\|$.

(22) Let X be a non empty set, Y be a real normed space, f, g, h be points of the real normed space of bounded functions from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(23) Let X be a non empty set, Y be a real normed space, f, h be points of the real normed space of bounded functions from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $h' = h$. Let a be a real number. Then $h = a \cdot f$ if and only if for every element x of X holds $h'(x) = a \cdot f'(x)$.

(24) Let X be a non empty set, Y be a real normed space, f, g be points of the real normed space of bounded functions from X into Y, and a be a real number. Then

(i) $\|f\| = 0$ iff $f = 0_{\text{the real normed space of bounded functions from } X \text{ into } Y}$,

(ii) $\|a \cdot f\| = |a| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

(25) Let X be a non empty set and Y be a real normed space. Then the real normed space of bounded functions from X into Y is real normed space-like.

(26) Let X be a non empty set and Y be a real normed space. Then the real normed space of bounded functions from X into Y is a real normed space.

Let X be a non empty set and let Y be a real normed space. Observe that the real normed space of bounded functions from X into Y is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

We now state three propositions:

(27) Let X be a non empty set, Y be a real normed space, f, g, h be points of the real normed space of bounded functions from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f - g$ if and only if for every element x of X holds $h'(x) = f'(x) - g'(x)$.

(28) Let X be a non empty set and Y be a real normed space. Suppose Y is complete. Let s_1 be a sequence of the real normed space of bounded functions from X into Y. If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(29) Let X be a non empty set and Y be a real Banach space. Then the real normed space of bounded functions from X into Y is a real Banach space.

Let X be a non empty set and let Y be a real Banach space. One can verify that the real normed space of bounded functions from X into Y is complete.

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Solving Roots of Polynomial Equation of Degree 2 and 3 with Complex Coefficients

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Summary. In the article, solving complex roots of polynomial equation of degree 2 and 3 with real coefficients and complex roots of polynomial equation of degree 2 and 3 with complex coefficients is discussed.

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The terminology and notation used here are introduced in the following articles: [20], [15], [2], [5], [3], [8], [17], [16], [14], [10], [12], [7], [18], [1], [13], [21], [9], [19], [11], [6], and [4].

1. Solving Complex Roots of Polynomial Equation of Degree 2 and 3 with Real Coefficients

We follow the rules: a, b, c, d, a', b', c', d', x, y, x_1, u, v are real numbers and s, t, h, z, z_1, z_2, z_3, z_4, s_1, s_2, s_3, p, q are elements of $\mathbb{C}$.

Let a be a real number and let us consider z. Then $a \cdot z$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 1) $a \cdot z = (a + 0i) \cdot z$.

Then $a + z$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 2) $a + z = z + (a + 0i)$.

Let us consider z. Then $z^{\mathbf{2}}$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 3) $z^{\mathbf{2}} = (\Re(z)^{\mathbf{2}} - \Im(z)^{\mathbf{2}}) + (2 \cdot (\Re(z) \cdot \Im(z)))i$.

Let us consider a, b, c, z. Then $\mathrm{Poly2}(a, b, c, z)$ is an element of $\mathbb{C}$.

The following propositions are true:

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(1) $(a + ci) \cdot (b + di) = (a \cdot b - c \cdot d) + (a \cdot d + b \cdot c)i$.

(2) If $z = x + yi$, then $z^{\mathbf{2}} = (x^{\mathbf{2}} - y^{\mathbf{2}}) + (2 \cdot x \cdot y)i$.

(3) For all a, b holds $(a + 0i) \cdot (b + 0i) = a \cdot b + 0i$.

(4) If $a \neq 0$ and $\Delta(a, b, c) \geqslant 0$ and $\mathrm{Poly2}(a, b, c, z) = 0$, then $z = \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $z = \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $z = -\frac{b}{2 \cdot a}$.

(5) If $a \neq 0$ and $\Delta(a, b, c) < 0$ and $\mathrm{Poly2}(a, b, c, z) = 0_{\mathbb{C}}$, then $z = -\frac{b}{2 \cdot a} + \frac{\sqrt{-\Delta(a,b,c)}}{2 \cdot a}i$ or $z = -\frac{b}{2 \cdot a} + (-\frac{\sqrt{-\Delta(a,b,c)}}{2 \cdot a})i$.

(6) If $b \neq 0$ and for every z holds $\mathrm{Poly2}(0, b, c, z) = 0_{\mathbb{C}}$, then $z = -\frac{c}{b}$.

(7) Let a, b, c be real numbers and z, z_1, z_2 be elements of $\mathbb{C}$. Suppose $a \neq 0$. Suppose that for every element z of $\mathbb{C}$ holds $\mathrm{Poly2}(a, b, c, z) = \mathrm{Quard}(a, z_1, z_2, z)$. Then $\frac{b}{a} + 0i = -(z_1 + z_2)$ and $\frac{c}{a} + 0i = z_1 \cdot z_2$.

Let z be an element of $\mathbb{C}$. The functor $z^{\mathbf{3}}$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 4) $z^{\mathbf{3}} = z^{\mathbf{2}} \cdot z$.

Let a, b, c, d be real numbers and let z be an element of $\mathbb{C}$. The functor $\mathrm{Poly}_3(a, b, c, d, z)$ yielding an element of $\mathbb{C}$ is defined as follows:

(Def. 5) $\mathrm{Poly}_3(a, b, c, d, z) = a \cdot z^{\mathbf{3}} + b \cdot z^{\mathbf{2}} + c \cdot z + d$.

We now state a number of propositions:

(8) $(0_{\mathbb{C}})^{\mathbf{3}} = 0_{\mathbb{C}}$.

(9) $(1_{\mathbb{C}})^{\mathbf{3}} = 1_{\mathbb{C}}$.

(10) $(-1_{\mathbb{C}})^{\mathbf{3}} = -1_{\mathbb{C}}$.

(11) $\Re(z^{\mathbf{3}}) = \Re(z)^3 - 3 \cdot \Re(z) \cdot \Im(z)^{\mathbf{2}}$ and $\Im(z^{\mathbf{3}}) = -\Im(z)^3 + 3 \cdot \Re(z)^{\mathbf{2}} \cdot \Im(z)$.

(12) If for every z holds $\mathrm{Poly}_3(a, b, c, d, z) = \mathrm{Poly}_3(a', b', c', d', z)$, then $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$.

(13) $(z + h)^{\mathbf{3}} = z^{\mathbf{3}} + 3 \cdot h \cdot z^{\mathbf{2}} + 3 \cdot h^{\mathbf{2}} \cdot z + h^{\mathbf{3}}$.

(14) $(z \cdot h)^{\mathbf{3}} = z^{\mathbf{3}} \cdot h^{\mathbf{3}}$.

(15) If $b \neq 0$ and $\mathrm{Poly}_3(0, b, c, d, z) = 0_{\mathbb{C}}$ and $\Delta(b, c, d) \geqslant 0$, then $z = \frac{-c+\sqrt{\Delta(b,c,d)}}{2 \cdot b}$ or $z = \frac{-c-\sqrt{\Delta(b,c,d)}}{2 \cdot b}$ or $z = -\frac{c}{2 \cdot b}$.

(16) If $b \neq 0$ and $\mathrm{Poly}_3(0, b, c, d, z) = 0_{\mathbb{C}}$ and $\Delta(b, c, d) < 0$, then $z = -\frac{c}{2 \cdot b} + \frac{\sqrt{-\Delta(b,c,d)}}{2 \cdot b}i$ or $z = -\frac{c}{2 \cdot b} + (-\frac{\sqrt{-\Delta(b,c,d)}}{2 \cdot b})i$.

(17) If $a \neq 0$ and $\mathrm{Poly}_3(a, 0, c, 0, z) = 0$ and $4 \cdot a \cdot c \leqslant 0$, then $z = \frac{\sqrt{-4 \cdot a \cdot c}}{2 \cdot a}$ or $z = \frac{-\sqrt{-4 \cdot a \cdot c}}{2 \cdot a}$ or $z = 0$.

(18) If $a \neq 0$ and $\mathrm{Poly}_3(a, b, c, 0, z) = 0$ and $\Delta(a, b, c) \geqslant 0$, then $z = \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $z = \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $z = -\frac{b}{2 \cdot a}$ or $z = 0$.

(19) If $a \neq 0$ and $\mathrm{Poly}_3(a, b, c, 0, z) = 0_{\mathbb{C}}$ and $\Delta(a, b, c) < 0$, then $z = -\frac{b}{2 \cdot a} + \frac{\sqrt{-\Delta(a,b,c)}}{2 \cdot a}i$ or $z = -\frac{b}{2 \cdot a} + (-\frac{\sqrt{-\Delta(a,b,c)}}{2 \cdot a})i$ or $z = 0_{\mathbb{C}}$.

(20) If $a \geqslant 0$ and $y^{\mathbf{2}} = a$, then $y = \sqrt{a}$ or $y = -\sqrt{a}$.

(21) Suppose $a \neq 0$ and $\mathrm{Poly}_3(a, 0, c, d, z) = 0_{\mathbb{C}}$ and $\Im(z) = 0$. Let given u, v. Suppose $\Re(z) = u + v$ and $3 \cdot v \cdot u + \frac{c}{a} = 0$. Then

(i) $z = \sqrt[3]{-\frac{d}{2 \cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}} + \sqrt[3]{-\frac{d}{2 \cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}}$, or

(ii) $z = \sqrt[3]{-\frac{d}{2 \cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}} + \sqrt[3]{-\frac{d}{2 \cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}}$, or

(iii) $z = \sqrt[3]{-\frac{d}{2 \cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}} + \sqrt[3]{-\frac{d}{2 \cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}}$.

(22) Suppose $a \neq 0$ and $\mathrm{Poly}_3(a, 0, c, d, z) = 0_{\mathbb{C}}$ and $\Im(z) \neq 0$. Let given u, v. Suppose $\Re(z) = u + v$ and $3 \cdot v \cdot u + \frac{c}{4 \cdot a} = 0$ and $\frac{c}{a} \geqslant 0$. Then

(i) $z = (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})} +$
$\sqrt{3 \cdot (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}}} + \frac{c}{a}} i$, or

(ii) $z = (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})} +$
$(-\sqrt{3 \cdot (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}}} + \frac{c}{a}})i$,
or

(iii) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} +$
$\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}}} + \frac{c}{a}} i$, or

(iv) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} +$
$(-\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}}} + \frac{c}{a}})i$, or

(v) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} +$
$\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}}} + \frac{c}{a}} i$, or

(vi) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} +$
$(-\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}}} + \frac{c}{a}})i$.

(23) Suppose $a \neq 0$ and $\mathrm{Poly}_3(a, b, c, d, z) = 0_{\mathbb{C}}$ and $\Im(z) = 0$. Let given u, v, x_1. Suppose $x_1 = \Re(z) + \frac{b}{3 \cdot a}$ and $\Re(z) = (u + v) - \frac{b}{3 \cdot a}$ and $3 \cdot u \cdot v + \frac{3 \cdot a \cdot c - b^{\mathbf{2}}}{3 \cdot a^{\mathbf{2}}} = 0$. Then

(i) $z = ((\sqrt[3]{(-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d - b\cdot c}{6\cdot a^2}) + \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3 + \frac{3\cdot a\cdot d - b\cdot c}{3\cdot a^2})^2}{4} + (\frac{3\cdot a\cdot c - b^2}{9\cdot a^2})^3}} + \sqrt[3]{-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d - b\cdot c}{6\cdot a^2} - \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3 + \frac{3\cdot a\cdot d - b\cdot c}{3\cdot a^2})^2}{4} + (\frac{3\cdot a\cdot c - b^2}{9\cdot a^2})^3}}) - \frac{b}{3\cdot a}) + 0i$, or

(ii) $z = ((\sqrt[3]{(-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d - b\cdot c}{6\cdot a^2}) + \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3 + \frac{3\cdot a\cdot d - b\cdot c}{3\cdot a^2})^2}{4} + (\frac{3\cdot a\cdot c - b^2}{9\cdot a^2})^3}} + \sqrt[3]{(-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d - b\cdot c}{6\cdot a^2}) + \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3 + \frac{3\cdot a\cdot d - b\cdot c}{3\cdot a^2})^2}{4} + (\frac{3\cdot a\cdot c - b^2}{9\cdot a^2})^3}}) - \frac{b}{3\cdot a}) + 0i$, or

(iii) $z = ((\sqrt[3]{-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d - b\cdot c}{6\cdot a^2} - \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3 + \frac{3\cdot a\cdot d - b\cdot c}{3\cdot a^2})^2}{4} + (\frac{3\cdot a\cdot c - b^2}{9\cdot a^2})^3}} + \sqrt[3]{-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d - b\cdot c}{6\cdot a^2} - \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3 + \frac{3\cdot a\cdot d - b\cdot c}{3\cdot a^2})^2}{4} + (\frac{3\cdot a\cdot c - b^2}{9\cdot a^2})^3}}) - \frac{b}{3\cdot a}) + 0i$.

(24) If $z_1 \neq 0$ and $\text{Poly1}(z_1, z_2, z) = 0$, then $z = -\frac{z_2}{z_1}$.

(25) If $z_2 \neq 0$, then it is not true that there exists z such that $\text{Poly1}(0, z_2, z) = 0$.

2. Complex Roots of Polynomial Equation of Degree 2 and 3 with Complex Coefficients

Let us consider z_1, z_2, z_3, z. The functor $\text{CPoly2}(z_1, z_2, z_3, z)$ yields an element of $\mathbb{C}$ and is defined by:

(Def. 6) $\text{CPoly2}(z_1, z_2, z_3, z) = z_1 \cdot z^2 + z_2 \cdot z + z_3$.

We now state a number of propositions:

(26) If for every z holds $\text{CPoly2}(z_1, z_2, z_3, z) = \text{CPoly2}(s_1, s_2, s_3, z)$, then $z_1 = s_1$ and $z_2 = s_2$ and $z_3 = s_3$.

(27) $\frac{-a+\sqrt{a^2+b^2}}{2} \geqslant 0$ and $\frac{a+\sqrt{a^2+b^2}}{2} \geqslant 0$.

(28) If $z^2 = s$ and $\Im(s) \geqslant 0$, then $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}}i$ or $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}})i$.

(29) If $z^2 = s$ and $\Im(s) = 0$ and $\Re(s) > 0$, then $z = \sqrt{\Re(s)}$ or $z = -\sqrt{\Re(s)}$.

(30) If $z^2 = s$ and $\Im(s) = 0$ and $\Re(s) < 0$, then $z = 0 + \sqrt{-\Re(s)}i$ or $z = 0 + (-\sqrt{-\Re(s)})i$.

(31) If $z^2 = s$ and $\Im(s) < 0$, then $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}})i$ or $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}}i$.

(32) Suppose $z^{\mathbf{2}} = s$. Then

(i) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$, or

(ii) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iii) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iv) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(33) $\mathrm{CPoly2}(0_{\mathbb{C}}, 0_{\mathbb{C}}, 0_{\mathbb{C}}, z) = 0$.

(34) If $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, 0_{\mathbb{C}}, 0_{\mathbb{C}}, z) = 0$, then $z = 0$.

(35) If $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, 0_{\mathbb{C}}, z) = 0$, then $z = -\frac{z_2}{z_1}$ or $z = 0$.

(36) Suppose $z_1 \neq 0_{\mathbb{C}}$ and $\mathrm{CPoly2}(z_1, 0_{\mathbb{C}}, z_3, z) = 0_{\mathbb{C}}$. Let given s. Suppose $s = -\frac{z_3}{z_1}$. Then

(i) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$, or

(ii) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iii) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iv) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(37) Suppose $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, z_3, z) = 0_{\mathbb{C}}$. Let given h, t. Suppose $h = (\frac{z_2}{2 \cdot z_1})^{\mathbf{2}} - \frac{z_3}{z_1}$ and $t = \frac{z_2}{2 \cdot z_1}$. Then

(i) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$, or

(ii) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(iii) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(iv) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$.

Let us consider z_1, z_2, z_3, z_4, z. The functor $\mathrm{CPoly2}(z_1, z_2, z_3, z_4, z)$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 7) $\mathrm{CPoly2}(z_1, z_2, z_3, z_4, z) = z_1 \cdot z^{\mathbf{3}} + z_2 \cdot z^{\mathbf{2}} + z_3 \cdot z + z_4$.

One can prove the following propositions:

(38) If $z^{\mathbf{2}} = 1$, then $z = 1$ or $z = -1$.

(39) $z_{\mathbb{N}}^3 = z \cdot z \cdot z$ and $z_{\mathbb{N}}^3 = z^{\mathbf{2}} \cdot z$ and $z_{\mathbb{N}}^3 = z^{\mathbf{3}}$.

(40) If $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, 0_{\mathbb{C}}, 0_{\mathbb{C}}, z) = 0_{\mathbb{C}}$, then $z = -\frac{z_2}{z_1}$ or $z = 0$.

(41) Suppose $z_1 \neq 0_{\mathbb{C}}$ and $\mathrm{CPoly2}(z_1, 0_{\mathbb{C}}, z_3, 0_{\mathbb{C}}, z) = 0_{\mathbb{C}}$. Let given s. Suppose $s = -\frac{z_3}{z_1}$. Then

(i) $z = 0_{\mathbb{C}}$, or

(ii) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$, or

(iii) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iv) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}})i$, or

(v) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^2+\Im(s)^2}}{2}}i$.

(42) Suppose $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, z_3, 0_{\mathbb{C}}, z) = 0_{\mathbb{C}}$. Let given s, h, t. Suppose $s = -\frac{z_3}{z_1}$ and $h = (\frac{z_2}{2 \cdot z_1})^2 - \frac{z_3}{z_1}$ and $t = \frac{z_2}{2 \cdot z_1}$. Then

(i) $z = 0$, or

(ii) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^2+\Im(h)^2}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^2+\Im(h)^2}}{2}}i) - t$, or

(iii) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^2+\Im(h)^2}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^2+\Im(h)^2}}{2}})i) - t$, or

(iv) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^2+\Im(h)^2}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^2+\Im(h)^2}}{2}})i) - t$, or

(v) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^2+\Im(h)^2}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^2+\Im(h)^2}}{2}}i) - t$.

(43) If $z = s - (\frac{1}{3} + 0i) \cdot z_1$, then $z^2 = s^2 + (-(\frac{2}{3} + 0i)) \cdot z_1 \cdot s + (\frac{1}{9} + 0i) \cdot {z_1}^2$.

(44) If $z = s - (\frac{1}{3} + 0i) \cdot z_1$, then $z^3 = ((s^3 - z_1 \cdot s^2) + (\frac{1}{3} + 0i) \cdot {z_1}^2 \cdot s) - (\frac{1}{27} + 0i) \cdot {z_1}^3$.

(45) Suppose $\mathrm{CPoly2}(1_{\mathbb{C}}, z_1, z_2, z_3, z) = 0_{\mathbb{C}}$. Let given p, q, s. Suppose $z = s - (\frac{1}{3} + 0i) \cdot z_1$ and $p = -(\frac{1}{3} + 0i) \cdot {z_1}^2 + z_2$ and $q = ((\frac{2}{27} + 0i) \cdot {z_1}^3 - (\frac{1}{3} + 0i) \cdot z_1 \cdot z_2) + z_3$. Then $\mathrm{CPoly2}(1_{\mathbb{C}}, 0_{\mathbb{C}}, p, q, s) = 0_{\mathbb{C}}$.

(46) For every element z of $\mathbb{C}$ holds $|z| \cdot \cos \operatorname{Arg} z + (|z| \cdot \sin \operatorname{Arg} z)i = (|z| + 0i) \cdot (\cos \operatorname{Arg} z + \sin \operatorname{Arg} z i)$.

(47) For every element z of $\mathbb{C}$ and for every natural number n holds $z_{\mathbb{N}}^{n+1} = (z_{\mathbb{N}}^{n}) \cdot z$.

(48) For every element z of $\mathbb{C}$ holds $z_{\mathbb{N}}^{1} = z$.

(49) For every element z of $\mathbb{C}$ holds $z_{\mathbb{N}}^{2} = z \cdot z$.

(50) For every natural number n such that $n > 0$ holds $0_{\mathbb{N}}^{n} = 0$.

(51) For all elements x, y of $\mathbb{C}$ and for every natural number n holds $(x \cdot y)_{\mathbb{N}}^{n} = (x_{\mathbb{N}}^{n}) \cdot y_{\mathbb{N}}^{n}$.

(52) For every real number x such that $x > 0$ and for every natural number n holds $(x + 0i)_{\mathbb{N}}^{n} = x^n + 0i$.

(53) For every real number x and for every natural number n holds $(\cos x + \sin x i)_{\mathbb{N}}^{n} = \cos(n \cdot x) + \sin(n \cdot x)i$.

(54) For every element z of $\mathbb{C}$ and for every natural number n such that $z \neq 0_{\mathbb{C}}$ or $n > 0$ holds $z_{\mathbb{N}}^{n} = |z|^n \cdot \cos(n \cdot \operatorname{Arg} z) + (|z|^n \cdot \sin(n \cdot \operatorname{Arg} z))i$.

(55) For all natural numbers n, k and for every real number x such that $n \neq 0$ holds $(\cos(\frac{x+2 \cdot \pi \cdot k}{n}) + \sin(\frac{x+2 \cdot \pi \cdot k}{n})i)_{\mathbb{N}}^{n} = \cos x + \sin x i$.

(56) Let z be an element of $\mathbb{C}$ and n, k be natural numbers. If $n \neq 0$, then $z = (\sqrt[n]{|z|} \cdot \cos(\frac{\operatorname{Arg} z+2 \cdot \pi \cdot k}{n}) + (\sqrt[n]{|z|} \cdot \sin(\frac{\operatorname{Arg} z+2 \cdot \pi \cdot k}{n}))i)_{\mathbb{N}}^{n}$.

Let z be an element of $\mathbb{C}$ and let n be a non empty natural number. An element of $\mathbb{C}$ is called a complex root of n, z if:

(Def. 8) $\mathrm{It}_{\mathbb{N}}^{n} = z$.

Next we state several propositions:

(57) Let z be an element of $\mathbb{C}$, n be a non empty natural number, and k be a natural number. Then $\sqrt[n]{|z|} \cdot \cos(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n}) + (\sqrt[n]{|z|} \cdot \sin(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n}))i$ is a complex root of n, z.

(58) For every element z of $\mathbb{C}$ and for every complex root v of 1, z holds $v = z$.

(59) For every non empty natural number n and for every complex root v of n, $0_{\mathbb{C}}$ holds $v = 0_{\mathbb{C}}$.

(60) Let n be a non empty natural number, z be an element of $\mathbb{C}$, and v be a complex root of n, z. If $v = 0_{\mathbb{C}}$, then $z = 0_{\mathbb{C}}$.

(61) Let n be a non empty natural number and k be a natural number. Then $\cos(\frac{2\cdot\pi\cdot k}{n}) + \sin(\frac{2\cdot\pi\cdot k}{n})i$ is a complex root of n, $1_{\mathbb{C}}$.

(62) For every natural number k holds $\cos(\frac{2\cdot\pi\cdot k}{3}) + \sin(\frac{2\cdot\pi\cdot k}{3})i$ is a complex root of 3, $1_{\mathbb{C}}$.

(63) For all elements z, s of $\mathbb{C}$ and for every natural number n such that $s \neq 0$ and $z \neq 0$ and $n \geqslant 1$ and $s_{\mathbb{N}}^{n} = z_{\mathbb{N}}^{n}$ holds $|s| = |z|$.

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Complex Linear Space and Complex Normed Space

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Summary. In this article, we introduce the notion of complex linear space and complex normed space.

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The articles [16], [7], [18], [1], [14], [13], [15], [8], [19], [4], [5], [2], [11], [17], [6], [10], [9], [3], and [12] provide the terminology and notation for this paper.

1. Complex Linear Space

We consider CLS structures as extensions of loop structure as systems
⟨ a carrier, a zero, an addition, an external multiplication ⟩,
where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, and the external multiplication is a function from $[:\mathbb{C}, \text{the carrier}:]$ into the carrier.

Let us observe that there exists a CLS structure which is non empty.

Let V be a CLS structure. A vector of V is an element of V.

Let V be a non empty CLS structure, let v be a vector of V, and let z be a Complex. The functor $z \cdot v$ yielding an element of V is defined as follows:

(Def. 1) $z \cdot v = (\text{the external multiplication of } V)(\langle z, v \rangle)$.

Let Z_1 be a non empty set, let O be an element of Z_1, let F be a binary operation on Z_1, and let G be a function from $[:\mathbb{C}, Z_1:]$ into Z_1. One can verify that $\langle Z_1, O, F, G \rangle$ is non empty.

Let I_1 be a non empty CLS structure. We say that I_1 is complex linear space-like if and only if the conditions (Def. 2) are satisfied.

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(Def. 2)(i) For every Complex z and for all vectors v, w of I_1 holds $z \cdot (v+w) = z \cdot v + z \cdot w$,

(ii) for all Complexes z_1, z_2 and for every vector v of I_1 holds $(z_1+z_2) \cdot v = z_1 \cdot v + z_2 \cdot v$,

(iii) for all Complexes z_1, z_2 and for every vector v of I_1 holds $(z_1 \cdot z_2) \cdot v = z_1 \cdot (z_2 \cdot v)$, and

(iv) for every vector v of I_1 holds $1_{\mathbb{C}} \cdot v = v$.

Let us observe that there exists a non empty CLS structure which is non empty, strict, Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

A complex linear space is an Abelian add-associative right zeroed right complementable complex linear space-like non empty CLS structure.

One can prove the following proposition

(1) Let V be a non empty CLS structure. Suppose that for all vectors v, w of V holds $v + w = w + v$ and for all vectors u, v, w of V holds $(u+v)+w = u+(v+w)$ and for every vector v of V holds $v+0_V = v$ and for every vector v of V there exists a vector w of V such that $v + w = 0_V$ and for every Complex z and for all vectors v, w of V holds $z \cdot (v+w) = z \cdot v + z \cdot w$ and for all Complexes z_1, z_2 and for every vector v of V holds $(z_1 + z_2) \cdot v = z_1 \cdot v + z_2 \cdot v$ and for all Complexes z_1, z_2 and for every vector v of V holds $(z_1 \cdot z_2) \cdot v = z_1 \cdot (z_2 \cdot v)$ and for every vector v of V holds $1_{\mathbb{C}} \cdot v = v$. Then V is a complex linear space.

We adopt the following convention: V, X, Y are complex linear spaces, u, v, v_1, v_2 are vectors of V, and z, z_1, z_2 are Complexes.

The following propositions are true:

(2) If $z = 0_{\mathbb{C}}$ or $v = 0_V$, then $z \cdot v = 0_V$.

(3) If $z \cdot v = 0_V$, then $z = 0_{\mathbb{C}}$ or $v = 0_V$.

(4) $-v = (-1_{\mathbb{C}}) \cdot v$.

(5) If $v = -v$, then $v = 0_V$.

(6) If $v + v = 0_V$, then $v = 0_V$.

(7) $z \cdot -v = (-z) \cdot v$.

(8) $z \cdot -v = -z \cdot v$.

(9) $(-z) \cdot -v = z \cdot v$.

(10) $z \cdot (v-u) = z \cdot v - z \cdot u$.

(11) $(z_1 - z_2) \cdot v = z_1 \cdot v - z_2 \cdot v$.

(12) If $z \neq 0$ and $z \cdot v = z \cdot u$, then $v = u$.

(13) If $v \neq 0_V$ and $z_1 \cdot v = z_2 \cdot v$, then $z_1 = z_2$.

(14) Let F, G be finite sequences of elements of the carrier of V. Suppose $\operatorname{len} F = \operatorname{len} G$ and for every natural number k and for every vector v of V

such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = z \cdot v$. Then $\sum F = z \cdot \sum G$.

(15) $z \cdot \sum(\varepsilon_{(\text{the carrier of } V)}) = 0_V$.

(16) $z \cdot \sum\langle v, u\rangle = z \cdot v + z \cdot u$.

(17) $z \cdot \sum\langle u, v_1, v_2\rangle = z \cdot u + z \cdot v_1 + z \cdot v_2$.

(18) $\sum\langle v, v\rangle = (2 + 0i) \cdot v$.

(19) $\sum\langle -v, -v\rangle = (-2 + 0i) \cdot v$.

(20) $\sum\langle v, v, v\rangle = (3 + 0i) \cdot v$.

2. Subspace and Cosets of Subspaces in Complex Linear Space

In the sequel V_1, V_2, V_3 are subsets of V.

Let us consider V, V_1. We say that V_1 is linearly closed if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) For all vectors v, u of V such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$, and

(ii) for every Complex z and for every vector v of V such that $v \in V_1$ holds $z \cdot v \in V_1$.

Next we state several propositions:

(21) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $0_V \in V_1$.

(22) If V_1 is linearly closed, then for every vector v of V such that $v \in V_1$ holds $-v \in V_1$.

(23) If V_1 is linearly closed, then for all vectors v, u of V such that $v \in V_1$ and $u \in V_1$ holds $v - u \in V_1$.

(24) $\{0_V\}$ is linearly closed.

(25) If the carrier of $V = V_1$, then V_1 is linearly closed.

(26) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v + u : v \in V_1 \ \wedge \ u \in V_2\}$, then V_3 is linearly closed.

(27) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider V. A complex linear space is said to be a subspace of V if it satisfies the conditions (Def. 4).

(Def. 4)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the zero of it = the zero of V,

(iii) the addition of it = (the addition of V)$\restriction[\!:$ the carrier of it, the carrier of it $:\!]$, and

(iv) the external multiplication of it = (the external multiplication of V)$\restriction[\!:\mathbb{C}$, the carrier of it $:\!]$.

We use the following convention: W, W_1, W_2 denote subspaces of V, x denotes a set, and w, w_1, w_2 denote vectors of W.

We now state a number of propositions:

(28) If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(29) If $x \in W$, then $x \in V$.

(30) w is a vector of V.

(31) $0_W = 0_V$.

(32) $0_{(W_1)} = 0_{(W_2)}$.

(33) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(34) If $w = v$, then $z \cdot w = z \cdot v$.

(35) If $w = v$, then $-v = -w$.

(36) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(37) $0_V \in W$.

(38) $0_{(W_1)} \in W_2$.

(39) $0_W \in V$.

(40) If $u \in W$ and $v \in W$, then $u + v \in W$.

(41) If $v \in W$, then $z \cdot v \in W$.

(42) If $v \in W$, then $-v \in W$.

(43) If $u \in W$ and $v \in W$, then $u - v \in W$.

In the sequel D denotes a non empty set, d_1 denotes an element of D, A denotes a binary operation on D, and M denotes a function from $[\!:\mathbb{C}, D:\!]$ into D.

Next we state several propositions:

(44) Suppose $V_1 = D$ and $d_1 = 0_V$ and $A =$ (the addition of V)$\restriction[\!: V_1, V_1 :\!]$ and $M =$ (the external multiplication of V)$\restriction[\!: \mathbb{C}, V_1 :\!]$. Then $\langle D, d_1, A, M \rangle$ is a subspace of V.

(45) V is a subspace of V.

(46) Let V, X be strict complex linear spaces. If V is a subspace of X and X is a subspace of V, then $V = X$.

(47) If V is a subspace of X and X is a subspace of Y, then V is a subspace of Y.

(48) If the carrier of $W_1 \subseteq$ the carrier of W_2, then W_1 is a subspace of W_2.

(49) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a subspace of W_2.

Let us consider V. Observe that there exists a subspace of V which is strict.

The following propositions are true:

(50) For all strict subspaces W_1, W_2 of V such that the carrier of $W_1 =$ the carrier of W_2 holds $W_1 = W_2$.

(51) For all strict subspaces W_1, W_2 of V such that for every v holds $v \in W_1$ iff $v \in W_2$ holds $W_1 = W_2$.

(52) Let V be a strict complex linear space and W be a strict subspace of V. If the carrier of W = the carrier of V, then $W = V$.

(53) Let V be a strict complex linear space and W be a strict subspace of V. If for every vector v of V holds $v \in W$ iff $v \in V$, then $W = V$.

(54) If the carrier of $W = V_1$, then V_1 is linearly closed.

(55) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists a strict subspace W of V such that V_1 = the carrier of W.

Let us consider V. The functor $\mathbf{0}_V$ yields a strict subspace of V and is defined by:

(Def. 5) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let us consider V. The functor Ω_V yields a strict subspace of V and is defined as follows:

(Def. 6) Ω_V = the CLS structure of V.

We now state several propositions:

(56) $\mathbf{0}_W = \mathbf{0}_V$.

(57) $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(58) $\mathbf{0}_W$ is a subspace of V.

(59) $\mathbf{0}_V$ is a subspace of W.

(60) $\mathbf{0}_{(W_1)}$ is a subspace of W_2.

(61) Every strict complex linear space V is a subspace of Ω_V.

Let us consider V and let us consider v, W. The functor $v + W$ yielding a subset of V is defined by:

(Def. 7) $v + W = \{v + u : u \in W\}$.

Let us consider V and let us consider W. A subset of V is called a coset of W if:

(Def. 8) There exists v such that it $= v + W$.

In the sequel B, C denote cosets of W.

The following propositions are true:

(62) $0_V \in v + W$ iff $v \in W$.

(63) $v \in v + W$.

(64) $0_V + W$ = the carrier of W.

(65) $v + \mathbf{0}_V = \{v\}$.

(66) $v + \Omega_V$ = the carrier of V.

(67) $0_V \in v + W$ iff $v + W$ = the carrier of W.

(68) $v \in W$ iff $v + W$ = the carrier of W.

(69) If $v \in W$, then $z \cdot v + W$ = the carrier of W.

(70) If $z \neq 0_{\mathbb{C}}$ and $z \cdot v + W$ = the carrier of W, then $v \in W$.

(71) $v \in W$ iff $-v + W$ = the carrier of W.

(72) $u \in W$ iff $v + W = v + u + W$.

(73) $u \in W$ iff $v + W = (v - u) + W$.

(74) $v \in u + W$ iff $u + W = v + W$.

(75) $v + W = -v + W$ iff $v \in W$.

(76) If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.

(77) If $u \in v + W$ and $u \in -v + W$, then $v \in W$.

(78) If $z \neq 1_{\mathbb{C}}$ and $z \cdot v \in v + W$, then $v \in W$.

(79) If $v \in W$, then $z \cdot v \in v + W$.

(80) $-v \in v + W$ iff $v \in W$.

(81) $u + v \in v + W$ iff $u \in W$.

(82) $v - u \in v + W$ iff $u \in W$.

(83) $u \in v + W$ iff there exists v_1 such that $v_1 \in W$ and $u = v + v_1$.

(84) $u \in v + W$ iff there exists v_1 such that $v_1 \in W$ and $u = v - v_1$.

(85) There exists v such that $v_1 \in v + W$ and $v_2 \in v + W$ iff $v_1 - v_2 \in W$.

(86) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v + v_1 = u$.

(87) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v - v_1 = u$.

(88) For all strict subspaces W_1, W_2 of V holds $v + W_1 = v + W_2$ iff $W_1 = W_2$.

(89) For all strict subspaces W_1, W_2 of V such that $v + W_1 = u + W_2$ holds $W_1 = W_2$.

(90) C is linearly closed iff $C =$ the carrier of W.

(91) For all strict subspaces W_1, W_2 of V and for every coset C_1 of W_1 and for every coset C_2 of W_2 such that $C_1 = C_2$ holds $W_1 = W_2$.

(92) $\{v\}$ is a coset of $\mathbf{0}_V$.

(93) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.

(94) The carrier of W is a coset of W.

(95) The carrier of V is a coset of Ω_V.

(96) If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of V.

(97) $0_V \in C$ iff $C =$ the carrier of W.

(98) $u \in C$ iff $C = u + W$.

(99) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u + v_1 = v$.

(100) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u - v_1 = v$.

(101) There exists C such that $v_1 \in C$ and $v_2 \in C$ iff $v_1 - v_2 \in W$.

(102) If $u \in B$ and $u \in C$, then $B = C$.

3. Complex Normed Space

We consider complex normed space structures as extensions of CLS structure as systems

⟨ a carrier, a zero, an addition, an external multiplication, a norm ⟩,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from [: $\mathbb{C}$, the carrier :] into the carrier, and the norm is a function from the carrier into $\mathbb{R}$.

Let us mention that there exists a complex normed space structure which is non empty.

In the sequel X is a non empty complex normed space structure and x is a point of X.

Let us consider X, x. The functor $\|x\|$ yielding a real number is defined by:

(Def. 9) $\|x\| = (\text{the norm of } X)(x)$.

Let I_1 be a non empty complex normed space structure. We say that I_1 is complex normed space-like if and only if:

(Def. 10) For all points x, y of I_1 and for every z holds $\|x\| = 0$ iff $x = 0_{(I_1)}$ and $\|z \cdot x\| = |z| \cdot \|x\|$ and $\|x + y\| \leqslant \|x\| + \|y\|$.

One can verify that there exists a non empty complex normed space structure which is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, right complementable, and strict.

A complex normed space is a complex normed space-like complex linear space-like Abelian add-associative right zeroed right complementable non empty complex normed space structure.

We follow the rules: C_3 is a complex normed space and x, y, w, g are points of C_3.

The following propositions are true:

(103) $\|0_{(C_3)}\| = 0$.

(104) $\|-x\| = \|x\|$.

(105) $\|x - y\| \leqslant \|x\| + \|y\|$.

(106) $0 \leqslant \|x\|$.

(107) $\|z_1 \cdot x + z_2 \cdot y\| \leqslant |z_1| \cdot \|x\| + |z_2| \cdot \|y\|$.

(108) $\|x - y\| = 0$ iff $x = y$.

(109) $\|x - y\| = \|y - x\|$.

(110) $\|x\| - \|y\| \leqslant \|x - y\|$.

(111) $|\|x\| - \|y\|| \leqslant \|x - y\|$.

(112) $\|x - w\| \leqslant \|x - y\| + \|y - w\|$.

(113) If $x \neq y$, then $\|x - y\| \neq 0$.

We adopt the following rules: S, S_1, S_2 are sequences of C_3, n, m are natural numbers, and r is a real number.

One can prove the following proposition

(114) There exists S such that $\operatorname{rng} S = \{0_{(C_3)}\}$.

In this article we present several logical schemes. The scheme *ExCNSSeq* deals with a complex normed space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExCLSSeq* deals with a complex linear space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

Let C_3 be a complex linear space and let S_1, S_2 be sequences of C_3. The functor $S_1 + S_2$ yielding a sequence of C_3 is defined by:

(Def. 11) For every n holds $(S_1 + S_2)(n) = S_1(n) + S_2(n)$.

Let C_3 be a complex linear space and let S_1, S_2 be sequences of C_3. The functor $S_1 - S_2$ yielding a sequence of C_3 is defined by:

(Def. 12) For every n holds $(S_1 - S_2)(n) = S_1(n) - S_2(n)$.

Let C_3 be a complex linear space, let S be a sequence of C_3, and let x be an element of C_3. The functor $S - x$ yielding a sequence of C_3 is defined by:

(Def. 13) For every n holds $(S - x)(n) = S(n) - x$.

Let C_3 be a complex linear space, let S be a sequence of C_3, and let us consider z. The functor $z \cdot S$ yields a sequence of C_3 and is defined as follows:

(Def. 14) For every n holds $(z \cdot S)(n) = z \cdot S(n)$.

Let us consider C_3 and let us consider S. We say that S is convergent if and only if:

(Def. 15) There exists g such that for every r such that $0 < r$ there exists m such that for every n such that $m \leqslant n$ holds $\|S(n) - g\| < r$.

The following four propositions are true:

(115) If S_1 is convergent and S_2 is convergent, then $S_1 + S_2$ is convergent.

(116) If S_1 is convergent and S_2 is convergent, then $S_1 - S_2$ is convergent.

(117) If S is convergent, then $S - x$ is convergent.

(118) If S is convergent, then $z \cdot S$ is convergent.

Let us consider C_3 and let us consider S. The functor $\|S\|$ yielding a sequence of real numbers is defined as follows:

(Def. 16) For every n holds $\|S\|(n) = \|S(n)\|$.

The following proposition is true

(119) If S is convergent, then $\|S\|$ is convergent.

Let us consider C_3 and let us consider S. Let us assume that S is convergent. The functor $\lim S$ yields a point of C_3 and is defined as follows:

(Def. 17) For every r such that $0 < r$ there exists m such that for every n such that $m \leqslant n$ holds $\|S(n) - \lim S\| < r$.

The following propositions are true:

(120) If S is convergent and $\lim S = g$, then $\|S - g\|$ is convergent and $\lim\|S - g\| = 0$.

(121) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 + S_2) = \lim S_1 + \lim S_2$.

(122) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 - S_2) = \lim S_1 - \lim S_2$.

(123) If S is convergent, then $\lim(S - x) = \lim S - x$.

(124) If S is convergent, then $\lim(z \cdot S) = z \cdot \lim S$.

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The Banach Algebra of Bounded Linear Operators

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Summary. In this article, the basic properties of Banach algebra are described. This algebra is defined as the set of all bounded linear operators from one normed space to another.

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The papers [21], [8], [23], [25], [24], [5], [7], [6], [19], [4], [1], [2], [18], [10], [22], [13], [3], [20], [16], [15], [9], [12], [11], [14], and [17] provide the terminology and notation for this paper.

Let X be a non empty set and let f, g be elements of X^X. Then $g \cdot f$ is an element of X^X.

One can prove the following propositions:

(1) Let X, Y, Z be real linear spaces, f be a linear operator from X into Y, and g be a linear operator from Y into Z. Then $g \cdot f$ is a linear operator from X into Z.

(2) Let X, Y, Z be real normed spaces, f be a bounded linear operator from X into Y, and g be a bounded linear operator from Y into Z. Then

(i) $g \cdot f$ is a bounded linear operator from X into Z, and

(ii) for every vector x of X holds $\|(g \cdot f)(x)\| \leqslant (\mathrm{BdLinOpsNorm}(Y, Z))(g) \cdot (\mathrm{BdLinOpsNorm}(X, Y))(f) \cdot \|x\|$ and $(\mathrm{BdLinOpsNorm}(X, Z))(g \cdot f) \leqslant (\mathrm{BdLinOpsNorm}(Y, Z))(g) \cdot (\mathrm{BdLinOpsNorm}(X, Y))(f)$.

Let X be a real normed space and let f, g be bounded linear operators from X into X. Then $g \cdot f$ is a bounded linear operator from X into X.

Let X be a real normed space and let f, g be elements of $\mathrm{BdLinOps}(X, X)$. The functor $f + g$ yields an element of $\mathrm{BdLinOps}(X, X)$ and is defined as follows:

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(Def. 1) $f + g = (\mathrm{Add\_}(\mathrm{BdLinOps}(X, X),$
$\mathrm{RVectorSpaceOfLinearOperators}(X, X)))(f, g)$.

Let X be a real normed space and let f, g be elements of $\mathrm{BdLinOps}(X, X)$. The functor $g \cdot f$ yielding an element of $\mathrm{BdLinOps}(X, X)$ is defined as follows:

(Def. 2) $g \cdot f = \mathrm{modetrans}(g, X, X) \cdot \mathrm{modetrans}(f, X, X)$.

Let X be a real normed space, let f be an element of $\mathrm{BdLinOps}(X, X)$, and let a be a real number. The functor $a \cdot f$ yields an element of $\mathrm{BdLinOps}(X, X)$ and is defined by:

(Def. 3) $a \cdot f = (\mathrm{Mult\_}(\mathrm{BdLinOps}(X, X),$
$\mathrm{RVectorSpaceOfLinearOperators}(X, X)))(a, f)$.

Let X be a real normed space. The functor $\mathrm{FuncMult}(X)$ yielding a binary operation on $\mathrm{BdLinOps}(X, X)$ is defined as follows:

(Def. 4) For all elements f, g of $\mathrm{BdLinOps}(X, X)$ holds $(\mathrm{FuncMult}(X))(f, g) = f \cdot g$.

The following proposition is true

(3) For every real normed space X holds $\mathrm{id}_{\text{the carrier of } X}$ is a bounded linear operator from X into X.

Let X be a real normed space. The functor $\mathrm{FuncUnit}(X)$ yields an element of $\mathrm{BdLinOps}(X, X)$ and is defined as follows:

(Def. 5) $\mathrm{FuncUnit}(X) = \mathrm{id}_{\text{the carrier of } X}$.

One can prove the following propositions:

(4) Let X be a real normed space and f, g, h be bounded linear operators from X into X. Then $h = f \cdot g$ if and only if for every vector x of X holds $h(x) = f(g(x))$.

(5) For every real normed space X and for all bounded linear operators f, g, h from X into X holds $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(6) Let X be a real normed space and f be a bounded linear operator from X into X. Then $f \cdot \mathrm{id}_{\text{the carrier of } X} = f$ and $\mathrm{id}_{\text{the carrier of } X} \cdot f = f$.

(7) For every real normed space X and for all elements f, g, h of $\mathrm{BdLinOps}(X, X)$ holds $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(8) For every real normed space X and for every element f of $\mathrm{BdLinOps}(X, X)$ holds $f \cdot \mathrm{FuncUnit}(X) = f$ and $\mathrm{FuncUnit}(X) \cdot f = f$.

(9) For every real normed space X and for all elements f, g, h of $\mathrm{BdLinOps}(X, X)$ holds $f \cdot (g + h) = f \cdot g + f \cdot h$.

(10) For every real normed space X and for all elements f, g, h of $\mathrm{BdLinOps}(X, X)$ holds $(g + h) \cdot f = g \cdot f + h \cdot f$.

(11) Let X be a real normed space, f, g be elements of $\mathrm{BdLinOps}(X, X)$, and a, b be real numbers. Then $(a \cdot b) \cdot (f \cdot g) = a \cdot f \cdot (b \cdot g)$.

(12) For every real normed space X and for all elements f, g of $\mathrm{BdLinOps}(X,X)$ and for every real number a holds $a \cdot (f \cdot g) = (a \cdot f) \cdot g$.

Let X be a real normed space. The functor $\mathrm{RingOfBoundedLinearOperators}(X)$ yielding a double loop structure is defined as follows:

(Def. 6) $\mathrm{RingOfBoundedLinearOperators}(X) = \langle \mathrm{BdLinOps}(X,X), \mathrm{Add\_}(\mathrm{BdLinOps}(X,X), \mathrm{RVectorSpaceOfLinearOperators}(X,X)), \mathrm{FuncMult}(X), \mathrm{FuncUnit}(X), \mathrm{Zero\_}(\mathrm{BdLinOps}(X,X), \mathrm{RVectorSpaceOfLinearOperators}(X,X))\rangle$.

Let X be a real normed space. Observe that $\mathrm{RingOfBoundedLinearOperators}(X)$ is non empty and strict.

One can prove the following propositions:

(13) Let X be a real normed space and x, y, z be elements of $\mathrm{RingOfBoundedLinearOperators}(X)$. Then $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{RingOfBoundedLinearOperators}(X)} = x$ and there exists an element t of $\mathrm{RingOfBoundedLinearOperators}(X)$ such that $x + t = 0_{\mathrm{RingOfBoundedLinearOperators}(X)}$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{RingOfBoundedLinearOperators}(X)} = x$ and $\mathbf{1}_{\mathrm{RingOfBoundedLinearOperators}(X)} \cdot x = x$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$.

(14) For every real normed space X holds $\mathrm{RingOfBoundedLinearOperators}(X)$ is a ring.

Let X be a real normed space. Note that $\mathrm{RingOfBoundedLinearOperators}(X)$ is Abelian, add-associative, right zeroed, right complementable, associative, left unital, right unital, and distributive.

Let X be a real normed space.

The functor $\mathrm{RAlgebraOfBoundedLinearOperators}(X)$ yielding an algebra structure is defined as follows:

(Def. 7) $\mathrm{RAlgebraOfBoundedLinearOperators}(X) = \langle \mathrm{BdLinOps}(X,X), \mathrm{FuncMult}(X), \mathrm{Add\_}(\mathrm{BdLinOps}(X,X), \mathrm{RVectorSpaceOfLinearOperators}(X,X)), \mathrm{Mult\_}(\mathrm{BdLinOps}(X,X), \mathrm{RVectorSpaceOfLinearOperators}(X,X)), \mathrm{FuncUnit}(X), \mathrm{Zero\_}(\mathrm{BdLinOps}(X,X), \mathrm{RVectorSpaceOfLinearOperators}(X,X))\rangle$.

Let X be a real normed space.

Observe that $\mathrm{RAlgebraOfBoundedLinearOperators}(X)$ is non empty and strict.

Next we state the proposition

(15) Let X be a real normed space, x, y, z be elements of $\mathrm{RAlgebraOfBoundedLinearOperators}(X)$, and a, b be real numbers. Then $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{RAlgebraOfBoundedLinearOperators}(X)} = x$ and there exists an element t of $\mathrm{RAlgebraOfBoundedLinearOperators}(X)$ such that $x + t = 0_{\mathrm{RAlgebraOfBoundedLinearOperators}(X)}$ and $(x \cdot y) \cdot z =$

$x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{RAlgebraOfBoundedLinearOperators}(X)} = x$ and $\mathbf{1}_{\mathrm{RAlgebraOfBoundedLinearOperators}(X)} \cdot x = x$ and $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(y+z) \cdot x = y \cdot x + z \cdot x$ and $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ and $a \cdot (x+y) = a \cdot x + a \cdot y$ and $(a+b) \cdot x = a \cdot x + b \cdot x$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$ and $(a \cdot b) \cdot (x \cdot y) = a \cdot x \cdot (b \cdot y)$.

A BL algebra is an Abelian add-associative right zeroed right complementable associative algebra-like non empty algebra structure.

The following proposition is true

(16) For every real normed space X holds RAlgebraOfBoundedLinearOperators(X) is a BL algebra.

One can check that l1-Space is complete.

Let us mention that l1-Space is non trivial.

One can verify that there exists a real Banach space which is non trivial.

One can prove the following propositions:

(17) For every non trivial real normed space X there exists a vector w of X such that $\|w\| = 1$.

(18) For every non trivial real normed space X holds (BdLinOpsNorm(X, X)) $(\mathrm{id}_{\text{the carrier of } X}) = 1$.

We introduce normed algebra structures which are extensions of algebra structure and normed structure and are systems

⟨ a carrier, a multiplication, an addition, an external multiplication, a unity, a zero, a norm ⟩,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from [: ℝ, the carrier :] into the carrier, the unity and the zero are elements of the carrier, and the norm is a function from the carrier into ℝ.

Let us mention that there exists a normed algebra structure which is non empty.

Let X be a real normed space.

The functor RNormedAlgebraOfBoundedLinearOperators(X) yields a normed algebra structure and is defined by:

(Def. 8) RNormedAlgebraOfBoundedLinearOperators(X) = ⟨BdLinOps(X, X), FuncMult(X), Add_(BdLinOps(X, X), RVectorSpaceOfLinearOperators (X, X)), Mult_(BdLinOps(X, X), RVectorSpaceOfLinearOperators(X, X)), FuncUnit(X), Zero_(BdLinOps(X, X), RVectorSpaceOfLinearOperators (X, X)), BdLinOpsNorm(X, X)⟩.

Let X be a real normed space. One can verify that RNormedAlgebraOfBoundedLinearOperators(X) is non empty and strict.

Next we state two propositions:

(19) Let X be a real normed space, x, y, z be elements of RNormedAlgebraOfBoundedLinearOperators(X), and a, b be real num-

bers. Then $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{RNormedAlgebraOfBoundedLinearOperators}(X)} = x$ and there exists an element t of RNormedAlgebraOfBoundedLinearOperators(X) such that $x + t = 0_{\mathrm{RNormedAlgebraOfBoundedLinearOperators}(X)}$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{RNormedAlgebraOfBoundedLinearOperators}(X)} = x$ and $\mathbf{1}_{\mathrm{RNormedAlgebraOfBoundedLinearOperators}(X)} \cdot x = x$ and $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(y+z) \cdot x = y \cdot x + z \cdot x$ and $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ and $(a \cdot b) \cdot (x \cdot y) = a \cdot x \cdot (b \cdot y)$ and $a \cdot (x+y) = a \cdot x + a \cdot y$ and $(a+b) \cdot x = a \cdot x + b \cdot x$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$ and $1 \cdot x = x$.

(20) Let X be a real normed space.
Then RNormedAlgebraOfBoundedLinearOperators(X) is real normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, algebra-like, and real linear space-like.

Let us observe that there exists a non empty normed algebra structure which is real normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, algebra-like, real linear space-like, and strict.

A normed algebra is a real normed space-like Abelian add-associative right zeroed right complementable associative algebra-like real linear space-like non empty normed algebra structure.

Let X be a real normed space.

Observe that RNormedAlgebraOfBoundedLinearOperators(X) is real normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, algebra-like, and real linear space-like.

Let X be a non empty normed algebra structure. We say that X is Banach Algebra-like1 if and only if:

(Def. 9) For all elements x, y of X holds $\|x \cdot y\| \leqslant \|x\| \cdot \|y\|$.

We say that X is Banach Algebra-like2 if and only if:

(Def. 10) $\|\mathbf{1}_X\| = 1$.

We say that X is Banach Algebra-like3 if and only if:

(Def. 11) For every real number a and for all elements x, y of X holds $a \cdot (x \cdot y) = x \cdot (a \cdot y)$.

Let X be a normed algebra. We say that X is Banach Algebra-like if and only if the condition (Def. 12) is satisfied.

(Def. 12) X is Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left unital, left distributive, and complete.

Let us mention that every normed algebra which is Banach Algebra-like is also Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left distributive, left unital, and complete and every normed algebra which is Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left distributive, left unital, and complete is also Banach Algebra-like.

Let X be a non trivial real Banach space.

Note that RNormedAlgebraOfBoundedLinearOperators(X) is Banach Algebra-like.

One can verify that there exists a normed algebra which is Banach Algebra-like.

A Banach algebra is a Banach Algebra-like normed algebra.

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Complex Linear Space of Complex Sequences

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Summary. In this article, we introduce a notion of complex linear space of complex sequence and complex unitary space.

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The notation and terminology used here are introduced in the following papers: [18], [21], [22], [17], [5], [6], [10], [3], [7], [16], [9], [12], [19], [4], [1], [11], [15], [14], [2], [20], [13], and [8].

1. Linear Space of Complex Sequence

The non empty set the set of complex sequences is defined by:

(Def. 1) For every set x holds $x \in$ the set of complex sequences iff x is a complex sequence.

Let z be a set. Let us assume that $z \in$ the set of complex sequences. The functor $\mathrm{id}_{\mathrm{seq}}(z)$ yields a complex sequence and is defined by:

(Def. 2) $\mathrm{id}_{\mathrm{seq}}(z) = z$.

Let z be a set. Let us assume that $z \in \mathbb{C}$. The functor $\mathrm{id}_{\mathbb{C}}(z)$ yielding a Complex is defined by:

(Def. 3) $\mathrm{id}_{\mathbb{C}}(z) = z$.

One can prove the following propositions:

(1) There exists a binary operation A_1 on the set of complex sequences such that

(i) for all elements a, b of the set of complex sequences holds $A_1(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$, and

(ii) A_1 is commutative and associative.

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(2) There exists a function f from $[: \mathbb{C}, \text{the set of complex sequences} :]$ into the set of complex sequences such that for all sets r, x if $r \in \mathbb{C}$ and $x \in$ the set of complex sequences, then $f(\langle r, x \rangle) = \mathrm{id}_{\mathbb{C}}(r)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The binary operation $\mathrm{add}_{\mathrm{seq}}$ on the set of complex sequences is defined as follows:

(Def. 4) For all elements a, b of the set of complex sequences holds $\mathrm{add}_{\mathrm{seq}}(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$.

The function $\mathrm{mult}_{\mathrm{seq}}$ from $[: \mathbb{C}, \text{the set of complex sequences} :]$ into the set of complex sequences is defined as follows:

(Def. 5) For all sets z, x such that $z \in \mathbb{C}$ and $x \in$ the set of complex sequences holds $\mathrm{mult}_{\mathrm{seq}}(\langle z, x \rangle) = \mathrm{id}_{\mathbb{C}}(z)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The element CZeroseq of the set of complex sequences is defined by:

(Def. 6) For every natural number n holds $(\mathrm{id}_{\mathrm{seq}}(\mathrm{CZeroseq}))(n) = 0_{\mathbb{C}}$.

One can prove the following propositions:

(3) For every complex sequence x holds $\mathrm{id}_{\mathrm{seq}}(x) = x$.

(4) For all vectors v, w of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $v + w = \mathrm{id}_{\mathrm{seq}}(v) + \mathrm{id}_{\mathrm{seq}}(w)$.

(5) For every Complex z and for every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $z \cdot v = z\ \mathrm{id}_{\mathrm{seq}}(v)$.

One can check that $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ is Abelian.

Next we state several propositions:

(6) For all vectors u, v, w of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(u + v) + w = u + (v + w)$.

(7) For every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $v + 0_{\langle \text{the set of complex sequences}, \mathrm{CZeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle} = v$.

(8) Let v be a vector of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$. Then there exists a vector w of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ such that $v + w = 0_{\langle \text{the set of complex sequences}, \mathrm{CZeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}} \rangle}$.

(9) For every Complex z and for all vectors v, w of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $z \cdot (v + w) = z \cdot v + z \cdot w$.

(10) For all Complexes z_1, z_2 and for every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(z_1 + z_2) \cdot v = z_1 \cdot v + z_2 \cdot v$.

(11) For all Complexes z_1, z_2 and for every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(z_1 \cdot z_2) \cdot v = z_1 \cdot (z_2 \cdot v)$.

(12) For every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $1_{\mathbb{C}} \cdot v = v$.

The complex linear space the linear space of complex sequences is defined as follows:

(Def. 7) The linear space of complex sequences = $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$.

Let X be a complex linear space and let X_1 be a subset of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Add\_}(X_1, X)$ yields a binary operation on X_1 and is defined by:

(Def. 8) $\mathrm{Add\_}(X_1, X) = (\text{the addition of } X) \restriction [: X_1, X_1 :]$.

Let X be a complex linear space and let X_1 be a subset of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Mult\_}(X_1, X)$ yields a function from $[: \mathbb{C}, X_1 :]$ into X_1 and is defined as follows:

(Def. 9) $\mathrm{Mult\_}(X_1, X) = (\text{the external multiplication of } X) \restriction [: \mathbb{C}, X_1 :]$.

Let X be a complex linear space and let X_1 be a subset of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Zero\_}(X_1, X)$ yielding an element of X_1 is defined by:

(Def. 10) $\mathrm{Zero\_}(X_1, X) = 0_X$.

One can prove the following proposition

(13) Let V be a complex linear space and V_1 be a subset of V. Suppose V_1 is linearly closed and non empty. Then $\langle V_1, \mathrm{Zero\_}(V_1, V), \mathrm{Add\_}(V_1, V), \mathrm{Mult\_}(V_1, V)\rangle$ is a subspace of V.

The subset the set of l2-complex sequences of the linear space of complex sequences is defined by the conditions (Def. 11).

(Def. 11)(i) The set of l2-complex sequences is non empty, and

(ii) for every set x holds $x \in$ the set of l2-complex sequences iff $x \in$ the set of complex sequences and $|\,\mathrm{id}_{\mathrm{seq}}(x)|\,|\,\mathrm{id}_{\mathrm{seq}}(x)|$ is summable.

One can prove the following propositions:

(14) The set of l2-complex sequences is linearly closed and the set of l2-complex sequences is non empty.

(15) $\langle$the set of l2-complex sequences, Zero_(the set of l2-complex sequences, the linear space of complex sequences), Add_(the set of l2-complex sequences, the linear space of complex sequences), Mult_(the set of l2-complex sequences, the linear space of complex sequences)$\rangle$ is a subspace of the linear space of complex sequences.

(16) $\langle$the set of l2-complex sequences, Zero_(the set of l2-complex sequences, the linear space of complex sequences), Add_(the set of l2-complex sequences, the linear space of complex sequences), Mult_(the set of l2-complex sequences, the linear space of complex sequences)$\rangle$ is a complex linear space.

(17)(i) The carrier of the linear space of complex sequences = the set of complex sequences,

(ii) for every set x holds x is an element of the linear space of complex sequences iff x is a complex sequence,

(iii) for every set x holds x is a vector of the linear space of complex sequences iff x is a complex sequence,

(iv) for every vector u of the linear space of complex sequences holds $u = \mathrm{id}_{\mathrm{seq}}(u)$,

(v) for all vectors u, v of the linear space of complex sequences holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$, and

(vi) for every Complex z and for every vector u of the linear space of complex sequences holds $z \cdot u = z\ \mathrm{id}_{\mathrm{seq}}(u)$.

2. Unitary Space with Complex Coefficient

We introduce complex unitary space structures which are extensions of CLS structure and are systems

$\langle$ a carrier, a zero, an addition, an external multiplication, a scalar product $\rangle$,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from $[\!:\mathbb{C}$, the carrier $:\!]$ into the carrier, and the scalar product is a function from $[\!:$ the carrier, the carrier $:\!]$ into $\mathbb{C}$.

Let us note that there exists a complex unitary space structure which is non empty and strict.

Let D be a non empty set, let Z be an element of D, let a be a binary operation on D, let m be a function from $[\!:\mathbb{C}, D:\!]$ into D, and let s be a function from $[\!:D, D:\!]$ into $\mathbb{C}$. Note that $\langle D, Z, a, m, s\rangle$ is non empty.

We adopt the following rules: X is a non empty complex unitary space structure, a, b are Complexes, and x, y are points of X.

Let us consider X and let us consider x, y. The functor $(x|y)$ yields a Complex and is defined by:

(Def. 12) $(x|y) = (\text{the scalar product of } X)(\langle x, y\rangle)$.

Let I_1 be a non empty complex unitary space structure. We say that I_1 is complex unitary space-like if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let x, y, w be points of I_1 and given a. Then $(x|x) = 0$ iff $x = 0_{(I_1)}$ and $0 \leqslant \Re((x|x))$ and $0 = \Im((x|x))$ and $(x|y) = \overline{(y|x)}$ and $((x+y)|w) = (x|w) + (y|w)$ and $((a \cdot x)|y) = a \cdot (x|y)$.

Let us note that there exists a non empty complex unitary space structure which is complex unitary space-like, complex linear space-like, Abelian, add-associative, right zeroed, right complementable, and strict.

A complex unitary space is a complex unitary space-like complex linear space-like Abelian add-associative right zeroed right complementable non empty complex unitary space structure.

We use the following convention: X is a complex unitary space and x, y, z, u, v are points of X.

Next we state a number of propositions:

(18) $(0_X|0_X) = 0$.

(19) $(x|(y+z)) = (x|y) + (x|z)$.

(20) $(x|(a \cdot y)) = \overline{a} \cdot (x|y)$.

(21) $((a \cdot x)|y) = (x|(\overline{a} \cdot y))$.

(22) $((a \cdot x + b \cdot y)|z) = a \cdot (x|z) + b \cdot (y|z)$.

(23) $(x|(a \cdot y + b \cdot z)) = \overline{a} \cdot (x|y) + \overline{b} \cdot (x|z)$.

(24) $((-x)|y) = (x|-y)$.

(25) $((-x)|y) = -(x|y)$.

(26) $(x|-y) = -(x|y)$.

(27) $((-x)|-y) = (x|y)$.

(28) $((x-y)|z) = (x|z) - (y|z)$.

(29) $(x|(y-z)) = (x|y) - (x|z)$.

(30) $((x-y)|(u-v)) = ((x|u) - (x|v) - (y|u)) + (y|v)$.

(31) $(0_X|x) = 0$.

(32) $(x|0_X) = 0$.

(33) $((x+y)|(x+y)) = (x|x) + (x|y) + (y|x) + (y|y)$.

(34) $((x+y)|(x-y)) = (((x|x) - (x|y)) + (y|x)) - (y|y)$.

(35) $((x-y)|(x-y)) = ((x|x) - (x|y) - (y|x)) + (y|y)$.

(36) $|(x|x)| = \Re((x|x))$.

(37) $|(x|y)| \leqslant \sqrt{|(x|x)|} \cdot \sqrt{|(y|y)|}$.

Let us consider X and let us consider x, y. We say that x, y are orthogonal if and only if:

(Def. 14) $(x|y) = 0$.

Let us note that the predicate x, y are orthogonal is symmetric.

We now state several propositions:

(38) If x, y are orthogonal, then x, $-y$ are orthogonal.

(39) If x, y are orthogonal, then $-x$, y are orthogonal.

(40) If x, y are orthogonal, then $-x$, $-y$ are orthogonal.

(41) x, 0_X are orthogonal.

(42) If x, y are orthogonal, then $((x+y)|(x+y)) = (x|x) + (y|y)$.

(43) If x, y are orthogonal, then $((x-y)|(x-y)) = (x|x) + (y|y)$.

Let us consider X, x. The functor $\|x\|$ yields a real number and is defined as follows:

(Def. 15) $\|x\| = \sqrt{|(x|x)|}$.

We now state several propositions:

(44) $\|x\| = 0$ iff $x = 0_X$.

(45) $\|a \cdot x\| = |a| \cdot \|x\|$.

(46) $0 \leqslant \|x\|$.

(47) $|(x|y)| \leqslant \|x\| \cdot \|y\|$.

(48) $\|x + y\| \leqslant \|x\| + \|y\|$.

(49) $\|-x\| = \|x\|$.

(50) $\|x\| - \|y\| \leqslant \|x - y\|$.

(51) $|\|x\| - \|y\|| \leqslant \|x - y\|$.

Let us consider X, x, y. The functor $\rho(x, y)$ yielding a real number is defined as follows:

(Def. 16) $\rho(x, y) = \|x - y\|$.

One can prove the following proposition

(52) $\rho(x, y) = \rho(y, x)$.

Let us consider X, x, y. Let us observe that the functor $\rho(x, y)$ is commutative.

We now state a number of propositions:

(53) $\rho(x, x) = 0$.

(54) $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

(55) $x \neq y$ iff $\rho(x, y) \neq 0$.

(56) $\rho(x, y) \geqslant 0$.

(57) $x \neq y$ iff $\rho(x, y) > 0$.

(58) $\rho(x, y) = \sqrt{|((x - y)|(x - y))|}$.

(59) $\rho(x + y, u + v) \leqslant \rho(x, u) + \rho(y, v)$.

(60) $\rho(x - y, u - v) \leqslant \rho(x, u) + \rho(y, v)$.

(61) $\rho(x - z, y - z) = \rho(x, y)$.

(62) $\rho(x - z, y - z) \leqslant \rho(z, x) + \rho(z, y)$.

We follow the rules: s_1, s_2, s_3, s_4 are sequences of X and k, n, m are natural numbers.

The scheme *Ex Seq in CUS* deals with a non empty complex unitary space structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

> There exists a sequence s_1 of $\mathcal{A}$ such that for every n holds $s_1(n) = \mathcal{F}(n)$

for all values of the parameters.

Let us consider X and let us consider s_1. The functor $-s_1$ yielding a sequence of X is defined by:

(Def. 17) For every n holds $(-s_1)(n) = -s_1(n)$.

Let us consider X, let us consider s_1, and let us consider x. The functor $s_1 + x$ yielding a sequence of X is defined by:

(Def. 18) For every n holds $(s_1 + x)(n) = s_1(n) + x$.

One can prove the following proposition

(63) $s_2 + s_3 = s_3 + s_2$.

Let us consider X, s_2, s_3. Let us observe that the functor $s_2 + s_3$ is commutative.

One can prove the following propositions:

(64) $s_2 + (s_3 + s_4) = (s_2 + s_3) + s_4$.

(65) If s_2 is constant and s_3 is constant and $s_1 = s_2 + s_3$, then s_1 is constant.

(66) If s_2 is constant and s_3 is constant and $s_1 = s_2 - s_3$, then s_1 is constant.

(67) If s_2 is constant and $s_1 = a \cdot s_2$, then s_1 is constant.

(68) s_1 is constant iff for every n holds $s_1(n) = s_1(n + 1)$.

(69) s_1 is constant iff for all n, k holds $s_1(n) = s_1(n + k)$.

(70) s_1 is constant iff for all n, m holds $s_1(n) = s_1(m)$.

(71) $s_2 - s_3 = s_2 + -s_3$.

(72) $s_1 = s_1 + 0_X$.

(73) $a \cdot (s_2 + s_3) = a \cdot s_2 + a \cdot s_3$.

(74) $(a + b) \cdot s_1 = a \cdot s_1 + b \cdot s_1$.

(75) $(a \cdot b) \cdot s_1 = a \cdot (b \cdot s_1)$.

(76) $1_{\mathbb{C}} \cdot s_1 = s_1$.

(77) $(-1_{\mathbb{C}}) \cdot s_1 = -s_1$.

(78) $s_1 - x = s_1 + -x$.

(79) $s_2 - s_3 = -(s_3 - s_2)$.

(80) $s_1 = s_1 - 0_X$.

(81) $s_1 = --s_1$.

(82) $s_2 - (s_3 + s_4) = s_2 - s_3 - s_4$.

(83) $(s_2 + s_3) - s_4 = s_2 + (s_3 - s_4)$.

(84) $s_2 - (s_3 - s_4) = (s_2 - s_3) + s_4$.

(85) $a \cdot (s_2 - s_3) = a \cdot s_2 - a \cdot s_3$.

3. Complex Unitary Space of Complex Sequence

Next we state the proposition

(86) There exists a function f from $[\!:$ the set of l2-complex sequences, the set of l2-complex sequences $:\!]$ into $\mathbb{C}$ such that for all sets x, y if $x \in$ the set of l2-complex sequences and $y \in$ the set of l2-complex sequences, then $f(\langle x, y \rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\, \overline{\mathrm{id}_{\mathrm{seq}}(y)})$.

The function $\mathrm{scalar}_{\mathrm{cl}}$ from $[\!:$ the set of l2-complex sequences, the set of l2-complex sequences $:\!]$ into $\mathbb{C}$ is defined by the condition (Def. 19).

(Def. 19) Let x, y be sets. Suppose $x \in$ the set of l2-complex sequences and $y \in$ the set of l2-complex sequences. Then $\mathrm{scalar}_{\mathrm{cl}}(\langle x, y \rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\, \overline{\mathrm{id}_{\mathrm{seq}}(y)})$.

Let us observe that $\langle$the set of l2-complex sequences, Zero_(the set of l2-complex sequences, the linear space of complex sequences), Add_(the set of l2-complex sequences, the linear space of complex sequences), Mult_(the set of l2-complex sequences, the linear space of complex sequences), $\mathrm{scalar}_{\mathrm{cl}}\rangle$ is non empty.

The non empty complex unitary space structure Complexl2-Space is defined by the condition (Def. 20).

(Def. 20) Complexl2-Space = $\langle$the set of l2-complex sequences, Zero_(the set of l2-complex sequences, the linear space of complex sequences), Add_(the set of l2-complex sequences, the linear space of complex sequences), Mult_(the set of l2-complex sequences, the linear space of complex sequences), $\mathrm{scalar}_{\mathrm{cl}}\rangle$.

The following propositions are true:

(87) Let l be a complex unitary space structure. Suppose $\langle$the carrier of l, the zero of l, the addition of l, the external multiplication of $l\rangle$ is a complex linear space. Then l is a complex linear space.

(88) For every complex sequence s_1 such that for every natural number n holds $s_1(n) = 0_{\mathbb{C}}$ holds s_1 is summable and $\sum s_1 = 0_{\mathbb{C}}$.

Let us observe that Complexl2-Space is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

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Behaviour of an Arc Crossing a Line

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Summary. In two-dimensional Euclidean space, we examine behaviour of an arc when it crosses a vertical line. There are three types when an arc enters into a line, which are: "Left-In", "Right-In" and "Oscillating-In". Also, there are three types when an arc goes out from a line, which are: "Left-Out", "Right-Out" and "Oscillating-Out". If an arc is a special polygonal arc, there are only two types for each case, entering in and going out. They are "Left-In" and "Right-In" for entering in, and "Left-Out" and "Right-Out" for going out.

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The articles [23], [26], [27], [7], [20], [16], [5], [15], [19], [24], [11], [6], [12], [9], [21], [10], [22], [2], [3], [14], [17], [18], [25], [4], [13], [1], and [8] provide the terminology and notation for this paper.

The following propositions are true:

(1) For every subset P of $\mathcal{E}^2_{\rm T}$ and for all points p_1, p_2, p of $\mathcal{E}^2_{\rm T}$ such that P is an arc from p_1 to p_2 and $p \in P$ holds $\mathrm{Segment}(P, p_1, p_2, p, p) = \{p\}$.

(2) For all points p_1, p_2, p of $\mathcal{E}^2_{\rm T}$ and for every real number a such that $p \in \mathcal{L}(p_1, p_2)$ and $(p_1)_{\mathbf{1}} \leqslant a$ and $(p_2)_{\mathbf{1}} \leqslant a$ holds $p_{\mathbf{1}} \leqslant a$.

(3) For all points p_1, p_2, p of $\mathcal{E}^2_{\rm T}$ and for every real number a such that $p \in \mathcal{L}(p_1, p_2)$ and $(p_1)_{\mathbf{1}} \geqslant a$ and $(p_2)_{\mathbf{1}} \geqslant a$ holds $p_{\mathbf{1}} \geqslant a$.

(4) For all points p_1, p_2, p of $\mathcal{E}^2_{\rm T}$ and for every real number a such that $p \in \mathcal{L}(p_1, p_2)$ and $(p_1)_{\mathbf{1}} < a$ and $(p_2)_{\mathbf{1}} < a$ holds $p_{\mathbf{1}} < a$.

(5) For all points p_1, p_2, p of $\mathcal{E}^2_{\rm T}$ and for every real number a such that $p \in \mathcal{L}(p_1, p_2)$ and $(p_1)_{\mathbf{1}} > a$ and $(p_2)_{\mathbf{1}} > a$ holds $p_{\mathbf{1}} > a$.

In the sequel j is a natural number.

Next we state two propositions:

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(6) Let f be a S-sequence in $\mathbb{R}^2$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $1 \leqslant j$ and $j < \operatorname{len} f$ and $p \in \mathcal{L}(f, j)$ and $q \in \mathcal{L}(f, j)$ and $(f_j)_{\mathbf{2}} = (f_{j+1})_{\mathbf{2}}$ and $(f_j)_{\mathbf{1}} > (f_{j+1})_{\mathbf{1}}$ and LE p, q, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\operatorname{len} f}$. Then $p_{\mathbf{1}} \geqslant q_{\mathbf{1}}$.

(7) Let f be a S-sequence in $\mathbb{R}^2$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $1 \leqslant j$ and $j < \operatorname{len} f$ and $p \in \mathcal{L}(f, j)$ and $q \in \mathcal{L}(f, j)$ and $(f_j)_{\mathbf{2}} = (f_{j+1})_{\mathbf{2}}$ and $(f_j)_{\mathbf{1}} < (f_{j+1})_{\mathbf{1}}$ and LE p, q, $\widetilde{\mathcal{L}}(f)$, f_1, $f_{\operatorname{len} f}$. Then $p_{\mathbf{1}} \leqslant q_{\mathbf{1}}$.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$, let p_1, p_2, p be points of $\mathcal{E}^2_{\mathrm{T}}$, and let e be a real number. We say that p is LIn of P, p_1, p_2, e if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) P is an arc from p_1 to p_2,
(ii) $p \in P$,
(iii) $p_{\mathbf{1}} = e$, and
(iv) there exists a point p_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_4)_{\mathbf{1}} < e$ and LE p_4, p, P, p_1, p_2 and for every point p_5 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_4, p_5, P, p_1, p_2 and LE p_5, p, P, p_1, p_2 holds $(p_5)_{\mathbf{1}} \leqslant e$.

We say that p is RIn of P, p_1, p_2, e if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) P is an arc from p_1 to p_2,
(ii) $p \in P$,
(iii) $p_{\mathbf{1}} = e$, and
(iv) there exists a point p_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_4)_{\mathbf{1}} > e$ and LE p_4, p, P, p_1, p_2 and for every point p_5 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_4, p_5, P, p_1, p_2 and LE p_5, p, P, p_1, p_2 holds $(p_5)_{\mathbf{1}} \geqslant e$.

We say that p is LOut of P, p_1, p_2, e if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) P is an arc from p_1 to p_2,
(ii) $p \in P$,
(iii) $p_{\mathbf{1}} = e$, and
(iv) there exists a point p_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_4)_{\mathbf{1}} < e$ and LE p, p_4, P, p_1, p_2 and for every point p_5 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_5, p_4, P, p_1, p_2 and LE p, p_5, P, p_1, p_2 holds $(p_5)_{\mathbf{1}} \leqslant e$.

We say that p is ROut of P, p_1, p_2, e if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) P is an arc from p_1 to p_2,
(ii) $p \in P$,
(iii) $p_{\mathbf{1}} = e$, and
(iv) there exists a point p_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that $(p_4)_{\mathbf{1}} > e$ and LE p, p_4, P, p_1, p_2 and for every point p_5 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_5, p_4, P, p_1, p_2 and LE p, p_5, P, p_1, p_2 holds $(p_5)_{\mathbf{1}} \geqslant e$.

We say that p is OsIn of P, p_1, p_2, e if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) P is an arc from p_1 to p_2,

(ii) $p \in P$,

(iii) $p_{\mathbf{1}} = e$, and

(iv) there exists a point p_7 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_7, p, P, p_1, p_2 and for every point p_8 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_7, p_8, P, p_1, p_2 and LE p_8, p, P, p_1, p_2 holds $(p_8)_{\mathbf{1}} = e$ and for every point p_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_4, p_7, P, p_1, p_2 and $p_4 \neq p_7$ holds there exists a point p_5 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_4, p_5, P, p_1, p_2 and LE p_5, p_7, P, p_1, p_2 and $(p_5)_{\mathbf{1}} > e$ and there exists a point p_6 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_4, p_6, P, p_1, p_2 and LE p_6, p_7, P, p_1, p_2 and $(p_6)_{\mathbf{1}} < e$.

We say that p is OsOut of P, p_1, p_2, e if and only if the conditions (Def. 6) are satisfied.

(Def. 6)(i) P is an arc from p_1 to p_2,

(ii) $p \in P$,

(iii) $p_{\mathbf{1}} = e$, and

(iv) there exists a point p_7 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p, p_7, P, p_1, p_2 and for every point p_8 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_8, p_7, P, p_1, p_2 and LE p, p_8, P, p_1, p_2 holds $(p_8)_{\mathbf{1}} = e$ and for every point p_4 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_7, p_4, P, p_1, p_2 and $p_4 \neq p_7$ holds there exists a point p_5 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_5, p_4, P, p_1, p_2 and LE p_7, p_5, P, p_1, p_2 and $(p_5)_{\mathbf{1}} > e$ and there exists a point p_6 of $\mathcal{E}^2_{\mathrm{T}}$ such that LE p_6, p_4, P, p_1, p_2 and LE p_7, p_6, P, p_1, p_2 and $(p_6)_{\mathbf{1}} < e$.

We now state a number of propositions:

(8) Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$, p_1, p_2, p be points of $\mathcal{E}^2_{\mathrm{T}}$, and e be a real number. Suppose P is an arc from p_1 to p_2 and $(p_1)_{\mathbf{1}} \leqslant e$ and $(p_2)_{\mathbf{1}} \geqslant e$. Then there exists a point p_3 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_3 \in P$ and $(p_3)_{\mathbf{1}} = e$.

(9) Let P be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, p_1, p_2, p be points of $\mathcal{E}^2_{\mathrm{T}}$, and e be a real number. Suppose P is an arc from p_1 to p_2 and $(p_1)_{\mathbf{1}} < e$ and $(p_2)_{\mathbf{1}} > e$ and $p \in P$ and $p_{\mathbf{1}} = e$. Then p is LIn of P, p_1, p_2, e, RIn of P, p_1, p_2, e, and OsIn of P, p_1, p_2, e.

(10) Let P be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, p_1, p_2, p be points of $\mathcal{E}^2_{\mathrm{T}}$, and e be a real number. Suppose P is an arc from p_1 to p_2 and $(p_1)_{\mathbf{1}} < e$ and $(p_2)_{\mathbf{1}} > e$ and $p \in P$ and $p_{\mathbf{1}} = e$. Then p is LOut of P, p_1, p_2, e, ROut of P, p_1, p_2, e, and OsOut of P, p_1, p_2, e.

(11) For every subset P of $\mathbb{I}$ and for every real number s such that $P = [0, s[$ holds P is open.

(12) For every subset P of $\mathbb{I}$ and for every real number s such that $P =]s, 1]$ holds P is open.

(13) Let P be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, P_1 be a subset of $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$, Q be a subset of $\mathbb{I}$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$, and s be a real number. Suppose

$s \leqslant 1$ and $P_1 = \{q_0; q_0$ ranges over points of $\mathcal{E}^2_{\rm T}$: $\bigvee_{s_1 : \text{real number}} (0 \leqslant s_1 \ \wedge\ s_1 < s \ \wedge\ q_0 = f(s_1))\}$ and $Q = [0, s[$. Then $f^\circ Q = P_1$.

(14) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, P_1 be a subset of $(\mathcal{E}^2_{\rm T})\restriction P$, Q be a subset of $\mathbb{I}$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$, and s be a real number. Suppose $s \geqslant 0$ and $P_1 = \{q_0; q_0$ ranges over points of $\mathcal{E}^2_{\rm T}$: $\bigvee_{s_1 : \text{real number}} (s < s_1 \ \wedge\ s_1 \leqslant 1 \ \wedge\ q_0 = f(s_1))\}$ and $Q =]s, 1]$. Then $f^\circ Q = P_1$.

(15) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, P_1 be a subset of $(\mathcal{E}^2_{\rm T})\restriction P$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$, and s be a real number. Suppose $s \leqslant 1$ and f is a homeomorphism and $P_1 = \{q_0; q_0$ ranges over points of $\mathcal{E}^2_{\rm T}$: $\bigvee_{s_1 : \text{real number}} (0 \leqslant s_1 \ \wedge\ s_1 < s \ \wedge\ q_0 = f(s_1))\}$. Then P_1 is open.

(16) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, P_1 be a subset of $(\mathcal{E}^2_{\rm T})\restriction P$, f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$, and s be a real number. Suppose $s \geqslant 0$ and f is a homeomorphism and $P_1 = \{q_0; q_0$ ranges over points of $\mathcal{E}^2_{\rm T}$: $\bigvee_{s_1 : \text{real number}} (s < s_1 \ \wedge\ s_1 \leqslant 1 \ \wedge\ q_0 = f(s_1))\}$. Then P_1 is open.

(17) Let T be a non empty topological structure, Q_1, Q_2 be subsets of T, and p_1, p_2 be points of T. Suppose $Q_1 \cap Q_2 = \emptyset$ and $Q_1 \cup Q_2 =$ the carrier of T and $p_1 \in Q_1$ and $p_2 \in Q_2$ and Q_1 is open and Q_2 is open. Then it is not true that there exists a map P from $\mathbb{I}$ into T such that P is continuous and $P(0) = p_1$ and $P(1) = p_2$.

(18) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, Q be a subset of $(\mathcal{E}^2_{\rm T})\restriction P$, and p_1, p_2, q be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $q \in P$ and $q \neq p_1$ and $q \neq p_2$ and $Q = P \setminus \{q\}$. Then Q is not connected and it is not true that there exists a map R from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P\restriction Q$ such that R is continuous and $R(0) = p_1$ and $R(1) = p_2$.

(19) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$. Then LE q_1, q_2, P, p_1, p_2 or LE q_2, q_1, P, p_1, p_2.

(20) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$ and $p_1 \neq q_1$. Then $\mathrm{Segment}(P, p_1, p_2, p_1, q_1)$ is an arc from p_1 to q_1.

(21) Let n be a natural number, p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$, and P, P_1 be non empty subsets of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2 and P_1 is an arc from p_1 to p_2 and $P_1 \subseteq P$, then $P_1 = P$.

(22) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $q_1 \in P$ and $p_2 \neq q_1$. Then $\mathrm{Segment}(P, p_1, p_2, q_1, p_2)$ is an arc from q_1 to p_2.

(23) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2, q_3 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2 and LE q_2, q_3, P, p_1, p_2. Then $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) \cup \mathrm{Segment}(P, p_1, p_2, q_2, q_3) = \mathrm{Segment}(P, p_1, p_2, q_1, q_3)$.

(24) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2, q_3 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2 and LE q_2, q_3, P, p_1, p_2. Then $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) \cap \mathrm{Segment}(P, p_1, p_2, q_2, q_3) = \{q_2\}$.

(25) For every non empty subset P of $\mathcal{E}^2_{\rm T}$ and for all points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ such that P is an arc from p_1 to p_2 holds $\mathrm{Segment}(P, p_1, p_2, p_1, p_2) = P$.

(26) Let T be a non empty topological space, w_1, w_2, w_3 be points of T, and h_1, h_2 be maps from $\mathbb{I}$ into T. Suppose h_1 is continuous and $w_1 = h_1(0)$ and $w_2 = h_1(1)$ and h_2 is continuous and $w_2 = h_2(0)$ and $w_3 = h_2(1)$. Then there exists a map h_3 from $\mathbb{I}$ into T such that h_3 is continuous and $w_1 = h_3(0)$ and $w_3 = h_3(1)$.

(27) Let T be a non empty topological space, a, b, c be points of T, G_1 be a path from a to b, and G_2 be a path from b to c. Suppose G_1 is continuous and G_2 is continuous and $G_1(0) = a$ and $G_1(1) = b$ and $G_2(0) = b$ and $G_2(1) = c$. Then $G_1 + G_2$ is continuous and $(G_1 + G_2)(0) = a$ and $(G_1 + G_2)(1) = c$.

(28) Let P, Q_1 be non empty subsets of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and Q_1 is an arc from q_1 to q_2 and LE q_1, q_2, P, p_1, p_2 and $Q_1 \subseteq P$. Then $Q_1 = \mathrm{Segment}(P, p_1, p_2, q_1, q_2)$.

(29) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, p_1, p_2, q_1, q_2, p be points of $\mathcal{E}^2_{\rm T}$, and e be a real number. Suppose $(p_1)_{\mathbf{1}} < e$ and $(p_2)_{\mathbf{1}} > e$ and q_1 is LIn of P, p_1, p_2, e and $(q_2)_{\mathbf{1}} = e$ and $\mathcal{L}(q_1, q_2) \subseteq P$ and $p \in \mathcal{L}(q_1, q_2)$. Then p is LIn of P, p_1, p_2, e.

(30) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, p_1, p_2, q_1, q_2, p be points of $\mathcal{E}^2_{\rm T}$, and e be a real number. Suppose $(p_1)_{\mathbf{1}} < e$ and $(p_2)_{\mathbf{1}} > e$ and q_1 is RIn of P, p_1, p_2, e and $(q_2)_{\mathbf{1}} = e$ and $\mathcal{L}(q_1, q_2) \subseteq P$ and $p \in \mathcal{L}(q_1, q_2)$. Then p is RIn of P, p_1, p_2, e.

(31) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, p_1, p_2, q_1, q_2, p be points of $\mathcal{E}^2_{\rm T}$, and e be a real number. Suppose $(p_1)_{\mathbf{1}} < e$ and $(p_2)_{\mathbf{1}} > e$ and q_1 is LOut of P, p_1, p_2, e and $(q_2)_{\mathbf{1}} = e$ and $\mathcal{L}(q_1, q_2) \subseteq P$ and $p \in \mathcal{L}(q_1, q_2)$. Then p is LOut of P, p_1, p_2, e.

(32) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, p_1, p_2, q_1, q_2, p be points of $\mathcal{E}^2_{\rm T}$, and e be a real number. Suppose $(p_1)_{\mathbf{1}} < e$ and $(p_2)_{\mathbf{1}} > e$ and q_1 is ROut of P, p_1, p_2, e and $(q_2)_{\mathbf{1}} = e$ and $\mathcal{L}(q_1, q_2) \subseteq P$ and $p \in \mathcal{L}(q_1, q_2)$. Then p is ROut of P, p_1, p_2, e.

(33) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, p_1, p_2, p be points of $\mathcal{E}^2_{\rm T}$, and e be a real number. Suppose P is a special polygonal arc joining p_1 and p_2 and $(p_1)_{\mathbf{1}} < e$ and $(p_2)_{\mathbf{1}} > e$ and $p \in P$ and $p_{\mathbf{1}} = e$. Then p is LIn of P, p_1, p_2, e and RIn of P, p_1, p_2, e.

(34) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$, p_1, p_2, p be points of $\mathcal{E}^2_{\rm T}$, and e be a real number. Suppose P is a special polygonal arc joining p_1 and p_2 and

$(p_1)_{\mathbf{1}} < e$ and $(p_2)_{\mathbf{1}} > e$ and $p \in P$ and $p_{\mathbf{1}} = e$. Then p is LOut of P, p_1, p_2, e and ROut of P, p_1, p_2, e.

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Some Set Series in Finite Topological Spaces. Fundamental Concepts for Image Processing

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Summary. First we give a definition of "inflation" of a set in finite topological spaces. Then a concept of "deflation" of a set is also defined. In the remaining part, we give a concept of the "set series" for a subset of a finite topological space. Using this, we can define a series of neighbourhoods for each point in the space. The work is done according to [7].

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The articles [9], [5], [10], [2], [8], [1], [12], [11], [3], [4], and [6] provide the notation and terminology for this paper.

We adopt the following rules: T denotes a non empty finite topology space, A, B denote subsets of T, and x, y denote elements of T.

Let us consider T and let A be a subset of T. The functor A^d yields a subset of T and is defined by:

(Def. 1) $A^d = \{x; x$ ranges over elements of T: $\bigwedge_{y\,:\,\text{element of } T} (y \in A^{\rm c} \Rightarrow x \notin U(y))\}$.

We now state a number of propositions:

(1) If T is filled, then $A \subseteq A^f$.

(2) $x \in A^d$ iff for every y such that $y \in A^{\rm c}$ holds $x \notin U(y)$.

(3) If T is filled, then $A^d \subseteq A$.

(4) $A^d = ((A^{\rm c})^f)^{\rm c}$.

(5) If $A \subseteq B$, then $A^f \subseteq B^f$.

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(6) If $A \subseteq B$, then $A^d \subseteq B^d$.

(7) $(A \cap B)^b \subseteq A^b \cap B^b$.

(8) $(A \cup B)^b = A^b \cup B^b$.

(9) $A^i \cup B^i \subseteq (A \cup B)^i$.

(10) $A^i \cap B^i = (A \cap B)^i$.

(11) $A^f \cup B^f = (A \cup B)^f$.

(12) $A^d \cap B^d = A \cap B^d$.

Let T be a non empty finite topology space and let A be a subset of T. The functor $\mathrm{Fcl}(A)$ yields a function from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ and is defined as follows:

(Def. 2) For every natural number n and for every subset B of T such that $B = (\mathrm{Fcl}(A))(n)$ holds $(\mathrm{Fcl}(A))(n+1) = B^b$ and $(\mathrm{Fcl}(A))(0) = A$.

Let T be a non empty finite topology space, let A be a subset of T, and let n be a natural number. The functor $\mathrm{Fcl}(A, n)$ yields a subset of T and is defined by:

(Def. 3) $\mathrm{Fcl}(A, n) = (\mathrm{Fcl}(A))(n)$.

Let T be a non empty finite topology space and let A be a subset of T. The functor $\mathrm{Fint}(A)$ yields a function from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ and is defined by:

(Def. 4) For every natural number n and for every subset B of T such that $B = (\mathrm{Fint}(A))(n)$ holds $(\mathrm{Fint}(A))(n+1) = B^i$ and $(\mathrm{Fint}(A))(0) = A$.

Let T be a non empty finite topology space, let A be a subset of T, and let n be a natural number. The functor $\mathrm{Fint}(A, n)$ yields a subset of T and is defined as follows:

(Def. 5) $\mathrm{Fint}(A, n) = (\mathrm{Fint}(A))(n)$.

The following propositions are true:

(13) For every natural number n holds $\mathrm{Fcl}(A, n+1) = (\mathrm{Fcl}(A, n))^b$.

(14) $\mathrm{Fcl}(A, 0) = A$.

(15) $\mathrm{Fcl}(A, 1) = A^b$.

(16) $\mathrm{Fcl}(A, 2) = (A^b)^b$.

(17) For every natural number n holds $\mathrm{Fcl}(A \cup B, n) = \mathrm{Fcl}(A, n) \cup \mathrm{Fcl}(B, n)$.

(18) For every natural number n holds $\mathrm{Fint}(A, n+1) = (\mathrm{Fint}(A, n))^i$.

(19) $\mathrm{Fint}(A, 0) = A$.

(20) $\mathrm{Fint}(A, 1) = A^i$.

(21) $\mathrm{Fint}(A, 2) = (A^i)^i$.

(22) For every natural number n holds $\mathrm{Fint}(A \cap B, n) = \mathrm{Fint}(A, n) \cap \mathrm{Fint}(B, n)$.

(23) If T is filled, then for every natural number n holds $A \subseteq \mathrm{Fcl}(A, n)$.

(24) If T is filled, then for every natural number n holds $\mathrm{Fint}(A, n) \subseteq A$.

(25) If T is filled, then for every natural number n holds $\mathrm{Fcl}(A, n) \subseteq \mathrm{Fcl}(A, n+1)$.

(26) If T is filled, then for every natural number n holds $\mathrm{Fint}(A, n+1) \subseteq \mathrm{Fint}(A, n)$.

(27) For every natural number n holds $(\mathrm{Fint}(A^{\mathrm{c}}, n))^{\mathrm{c}} = \mathrm{Fcl}(A, n)$.

(28) For every natural number n holds $(\mathrm{Fcl}(A^{\mathrm{c}}, n))^{\mathrm{c}} = \mathrm{Fint}(A, n)$.

(29) For every natural number n holds $\mathrm{Fcl}(A, n) \cup \mathrm{Fcl}(B, n) = (\mathrm{Fint}((A \cup B)^{\mathrm{c}}, n))^{\mathrm{c}}$.

(30) For every natural number n holds $\mathrm{Fint}(A, n) \cap \mathrm{Fint}(B, n) = (\mathrm{Fcl}((A \cap B)^{\mathrm{c}}, n))^{\mathrm{c}}$.

Let T be a non empty finite topology space and let A be a subset of T. The functor $\mathrm{Finf}(A)$ yielding a function from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ is defined by:

(Def. 6) For every natural number n and for every subset B of T such that $B = (\mathrm{Finf}(A))(n)$ holds $(\mathrm{Finf}(A))(n+1) = B^f$ and $(\mathrm{Finf}(A))(0) = A$.

Let T be a non empty finite topology space, let A be a subset of T, and let n be a natural number. The functor $\mathrm{Finf}(A, n)$ yielding a subset of T is defined as follows:

(Def. 7) $\mathrm{Finf}(A, n) = (\mathrm{Finf}(A))(n)$.

Let T be a non empty finite topology space and let A be a subset of T. The functor $\mathrm{Fdfl}(A)$ yields a function from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ and is defined as follows:

(Def. 8) For every natural number n and for every subset B of T such that $B = (\mathrm{Fdfl}(A))(n)$ holds $(\mathrm{Fdfl}(A))(n+1) = B^d$ and $(\mathrm{Fdfl}(A))(0) = A$.

Let T be a non empty finite topology space, let A be a subset of T, and let n be a natural number. The functor $\mathrm{Fdfl}(A, n)$ yields a subset of T and is defined as follows:

(Def. 9) $\mathrm{Fdfl}(A, n) = (\mathrm{Fdfl}(A))(n)$.

Next we state a number of propositions:

(31) For every natural number n holds $\mathrm{Finf}(A, n+1) = (\mathrm{Finf}(A, n))^f$.

(32) $\mathrm{Finf}(A, 0) = A$.

(33) $\mathrm{Finf}(A, 1) = A^f$.

(34) $\mathrm{Finf}(A, 2) = (A^f)^f$.

(35) For every natural number n holds $\mathrm{Finf}(A \cup B, n) = \mathrm{Finf}(A, n) \cup \mathrm{Finf}(B, n)$.

(36) If T is filled, then for every natural number n holds $A \subseteq \mathrm{Finf}(A, n)$.

(37) If T is filled, then for every natural number n holds $\mathrm{Finf}(A, n) \subseteq \mathrm{Finf}(A, n+1)$.

(38) For every natural number n holds $\mathrm{Fdfl}(A, n+1) = \mathrm{Fdfl}(A, n)^d$.

(39) $\mathrm{Fdfl}(A, 0) = A$.

(40) $\mathrm{Fdfl}(A, 1) = A^d$.

(41) $\mathrm{Fdfl}(A, 2) = (A^d)^d$.

(42) For every natural number n holds $\mathrm{Fdfl}(A \cap B, n) = \mathrm{Fdfl}(A, n) \cap \mathrm{Fdfl}(B, n)$.

(43) If T is filled, then for every natural number n holds $\mathrm{Fdfl}(A, n) \subseteq A$.

(44) If T is filled, then for every natural number n holds $\mathrm{Fdfl}(A, n+1) \subseteq \mathrm{Fdfl}(A, n)$.

(45) For every natural number n holds $\mathrm{Fdfl}(A, n) = (\mathrm{Finf}(A^{\mathrm{c}}, n))^{\mathrm{c}}$.

(46) For every natural number n holds $\mathrm{Fdfl}(A, n) \cap \mathrm{Fdfl}(B, n) = (\mathrm{Finf}((A \cap B)^{\mathrm{c}}, n))^{\mathrm{c}}$.

Let T be a non empty finite topology space, let n be a natural number, and let x be an element of T. The functor $U(x, n)$ yields a subset of T and is defined as follows:

(Def. 10) $U(x, n) = \mathrm{Finf}(U(x), n)$.

Next we state two propositions:

(47) $U(x, 0) = U(x)$.

(48) For every natural number n holds $U(x, n+1) = (U(x, n))^f$.

Let S, T be non empty finite topology spaces. We say that S, T are mutually symmetric if and only if the conditions (Def. 11) are satisfied.

(Def. 11)(i) The carrier of S = the carrier of T, and

(ii) for all sets x, y such that $x \in$ the carrier of S and $y \in$ the carrier of T holds $y \in$ (the neighbour-map of S)(x) iff $x \in$ (the neighbour-map of T)(y).

Let us note that the predicate S, T are mutually symmetric is symmetric.

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The Series on Banach Algebra

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Summary. In this article, the basic properties of the series on Banach algebra are described. The Neumann series is introduced in the last section.

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The notation and terminology used in this paper are introduced in the following articles: [19], [21], [22], [4], [5], [3], [2], [18], [6], [1], [20], [10], [11], [12], [17], [9], [7], [8], [14], [13], [15], and [16].

1. Basic Properties of Sequences of Norm Space

Let X be a non empty normed structure and let s_1 be a sequence of X. The functor $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ yielding a sequence of X is defined as follows:

(Def. 1) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(0) = s_1(0)$ and for every natural number n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + s_1(n+1)$.

One can prove the following proposition

(1) Let X be an add-associative right zeroed right complementable non empty normed structure and s_1 be a sequence of X. Suppose that for every natural number n holds $s_1(n) = 0_X$. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) = 0_X$.

Let X be a real normed space and let s_1 be a sequence of X. We say that s_1 is summable if and only if:

(Def. 2) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

Let X be a real normed space. One can verify that there exists a sequence of X which is summable.

Let X be a real normed space and let s_1 be a sequence of X. The functor $\sum s_1$ yields an element of X and is defined by:

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(Def. 3) $\sum s_1 = \lim((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}})$.

Let X be a real normed space and let s_1 be a sequence of X. We say that s_1 is norm-summable if and only if:

(Def. 4) $\|s_1\|$ is summable.

Next we state several propositions:

(2) For every real normed space X and for every sequence s_1 of X and for every natural number m holds $0 \leqslant \|s_1\|(m)$.

(3) For every real normed space X and for all elements x, y, z of X holds $\|x - y\| = \|(x - z) + (z - y)\|$.

(4) Let X be a real normed space and s_1 be a sequence of X. Suppose s_1 is convergent. Let s be a real number. Suppose $0 < s$. Then there exists a natural number n such that for every natural number m if $n \leqslant m$, then $\|s_1(m) - s_1(n)\| < s$.

(5) Let X be a real normed space and s_1 be a sequence of X. Then s_1 is Cauchy sequence by norm if and only if for every real number p such that $p > 0$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\|s_1(m) - s_1(n)\| < p$.

(6) Let X be a real normed space and s_1 be a sequence of X. Suppose that for every natural number n holds $s_1(n) = 0_X$. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa \in \mathbb{N}}(m) = 0$.

Let X be a real normed space and let s_1 be a sequence of X. Let us observe that s_1 is constant if and only if:

(Def. 5) There exists an element r of X such that for every natural number n holds $s_1(n) = r$.

Let X be a real normed space, let s_1 be a sequence of X, and let k be a natural number. The functor $s_1 \uparrow k$ yielding a sequence of X is defined as follows:

(Def. 6) For every natural number n holds $(s_1 \uparrow k)(n) = s_1(n + k)$.

Let X be a non empty 1-sorted structure, let N_1 be an increasing sequence of naturals, and let s_1 be a sequence of X. Then $s_1 \cdot N_1$ is a function from $\mathbb{N}$ into the carrier of X.

Let X be a non empty 1-sorted structure, let N_1 be an increasing sequence of naturals, and let s_1 be a sequence of X. Then $s_1 \cdot N_1$ is a sequence of X.

Let X be a real normed space and let s_1, s_2 be sequences of X. We say that s_1 is a subsequence of s_2 if and only if:

(Def. 7) There exists an increasing sequence N_1 of naturals such that $s_1 = s_2 \cdot N_1$.

Next we state a number of propositions:

(7) Let X be a non empty 1-sorted structure, s_1 be a sequence of X, N_1 be an increasing sequence of naturals, and n be a natural number. Then $(s_1 \cdot N_1)(n) = s_1(N_1(n))$.

(8) For every real normed space X and for every sequence s_1 of X holds $s_1 \uparrow 0 = s_1$.

(9) For every real normed space X and for every sequence s_1 of X and for all natural numbers k, m holds $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(10) For every real normed space X and for every sequence s_1 of X and for all natural numbers k, m holds $s_1 \uparrow k \uparrow m = s_1 \uparrow (k + m)$.

(11) Let X be a real normed space and s_1, s_2 be sequences of X. If s_2 is a subsequence of s_1 and s_1 is convergent, then s_2 is convergent.

(12) Let X be a real normed space and s_1, s_2 be sequences of X. If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(13) Let X be a real normed space, s_1 be a sequence of X, and k be a natural number. Then $s_1 \uparrow k$ is a subsequence of s_1.

(14) Let X be a real normed space, s_1, s_2 be sequences of X, and k be a natural number. If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(15) Let X be a real normed space and s_1, s_2 be sequences of X. Suppose s_1 is convergent and there exists a natural number k such that $s_1 = s_2 \uparrow k$. Then s_2 is convergent.

(16) Let X be a real normed space and s_1, s_2 be sequences of X. Suppose s_1 is convergent and there exists a natural number k such that $s_1 = s_2 \uparrow k$. Then $\lim s_2 = \lim s_1$.

(17) For every real normed space X and for every sequence s_1 of X such that s_1 is constant holds s_1 is convergent.

(18) Let X be a real normed space and s_1 be a sequence of X. If for every natural number n holds $s_1(n) = 0_X$, then s_1 is norm-summable.

Let X be a real normed space. Observe that there exists a sequence of X which is norm-summable.

Next we state three propositions:

(19) Let X be a real normed space and s be a sequence of X. If s is summable, then s is convergent and $\lim s = 0_X$.

(20) For every real normed space X and for all sequences s_3, s_4 of X holds $(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa}(s_4)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_3 + s_4)(\alpha))_{\kappa\in\mathbb{N}}$.

(21) For every real normed space X and for all sequences s_3, s_4 of X holds $(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa}(s_4)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_3 - s_4)(\alpha))_{\kappa\in\mathbb{N}}$.

Let X be a real normed space and let s_1 be a norm-summable sequence of X. Observe that $\|s_1\|$ is summable.

Let X be a real normed space. One can check that every sequence of X which is summable is also convergent.

The following propositions are true:

(22) Let X be a real normed space and s_2, s_5 be sequences of X. If s_2 is summable and s_5 is summable, then s_2+s_5 is summable and $\sum(s_2+s_5) = \sum s_2 + \sum s_5$.

(23) Let X be a real normed space and s_2, s_5 be sequences of X. If s_2 is summable and s_5 is summable, then s_2-s_5 is summable and $\sum(s_2-s_5) = \sum s_2 - \sum s_5$.

Let X be a real normed space and let s_2, s_5 be summable sequences of X. One can verify that $s_2 + s_5$ is summable and $s_2 - s_5$ is summable.

We now state two propositions:

(24) For every real normed space X and for every sequence s_1 of X and for every real number z holds $(\sum_{\alpha=0}^{\kappa}(z \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}} = z \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(25) Let X be a real normed space, s_1 be a summable sequence of X, and z be a real number. Then $z \cdot s_1$ is summable and $\sum(z \cdot s_1) = z \cdot \sum s_1$.

Let X be a real normed space, let z be a real number, and let s_1 be a summable sequence of X. Observe that $z \cdot s_1$ is summable.

One can prove the following two propositions:

(26) Let X be a real normed space and s, s_3 be sequences of X. If for every natural number n holds $s_3(n) = s(0)$, then $(\sum_{\alpha=0}^{\kappa}(s \uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}} \uparrow 1 - s_3$.

(27) Let X be a real normed space and s be a sequence of X. If s is summable, then for every natural number n holds $s \uparrow n$ is summable.

Let X be a real normed space, let s_1 be a summable sequence of X, and let n be a natural number. Observe that $s_1 \uparrow n$ is summable.

Next we state the proposition

(28) Let X be a real normed space and s_1 be a sequence of X. Then $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded if and only if s_1 is norm-summable.

Let X be a real normed space and let s_1 be a norm-summable sequence of X. One can check that $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded.

One can prove the following propositions:

(29) Let X be a real Banach space and s_1 be a sequence of X. Then s_1 is summable if and only if for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)\| < p$.

(30) Let X be a real normed space, s be a sequence of X, and n, m be natural numbers. If $n \leqslant m$, then $\|(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leqslant |(\sum_{\alpha=0}^{\kappa}\|s\|(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|s\|(\alpha))_{\kappa\in\mathbb{N}}(n)|$.

(31) For every real Banach space X and for every sequence s_1 of X such that s_1 is norm-summable holds s_1 is summable.

(32) Let X be a real normed space, r_1 be a sequence of real numbers, and s_5 be a sequence of X. Suppose r_1 is summable and there exists a natural

number m such that for every natural number n such that $m \leqslant n$ holds $\|s_5(n)\| \leqslant r_1(n)$. Then s_5 is norm-summable.

(33) Let X be a real normed space and s_2, s_5 be sequences of X. Suppose for every natural number n holds $0 \leqslant \|s_2\|(n)$ and $\|s_2\|(n) \leqslant \|s_5\|(n)$ and s_5 is norm-summable. Then s_2 is norm-summable and $\sum\|s_2\| \leqslant \sum\|s_5\|$.

(34) Let X be a real normed space and s_1 be a sequence of X. Suppose that
(i) for every natural number n holds $\|s_1\|(n) > 0$, and
(ii) there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\frac{\|s_1\|(n+1)}{\|s_1\|(n)} \geqslant 1$.
Then s_1 is not norm-summable.

(35) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose for every natural number n holds $r_1(n) = \sqrt[n]{\|s_1\|(n)}$ and r_1 is convergent and $\lim r_1 < 1$. Then s_1 is norm-summable.

(36) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose that
(i) for every natural number n holds $r_1(n) = \sqrt[n]{\|s_1\|(n)}$, and
(ii) there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $r_1(n) \geqslant 1$.
Then $\|s_1\|$ is not summable.

(37) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose for every natural number n holds $r_1(n) = \sqrt[n]{\|s_1\|(n)}$ and r_1 is convergent and $\lim r_1 > 1$. Then s_1 is not norm-summable.

(38) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose $\|s_1\|$ is non-increasing and for every natural number n holds $r_1(n) = 2^n \cdot \|s_1\|(2^n)$. Then s_1 is norm-summable if and only if r_1 is summable.

(39) Let X be a real normed space, s_1 be a sequence of X, and p be a real number. Suppose $p > 1$ and for every natural number n such that $n \geqslant 1$ holds $\|s_1\|(n) = \frac{1}{n^p}$. Then s_1 is norm-summable.

(40) Let X be a real normed space, s_1 be a sequence of X, and p be a real number. Suppose $p \leqslant 1$ and for every natural number n such that $n \geqslant 1$ holds $\|s_1\|(n) = \frac{1}{n^p}$. Then s_1 is not norm-summable.

(41) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose for every natural number n holds $s_1(n) \neq 0_X$ and $r_1(n) = \frac{\|s_1\|(n+1)}{\|s_1\|(n)}$ and r_1 is convergent and $\lim r_1 < 1$. Then s_1 is norm-summable.

(42) Let X be a real normed space and s_1 be a sequence of X. Suppose that
(i) for every natural number n holds $s_1(n) \neq 0_X$, and

(ii) there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\frac{\|s_1\|(n+1)}{\|s_1\|(n)} \geqslant 1$.

Then s_1 is not norm-summable.

Let X be a real Banach space. Observe that every sequence of X which is norm-summable is also summable.

2. Basic Properties of Sequences of Banach Algebra

The scheme *ExNCBASeq* deals with a non empty normed algebra structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The following proposition is true

(43) Let X be a Banach algebra, x, y, z be elements of X, and a, b be real numbers. Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x+0_X = x$ and there exists an element t of X such that $x+t = 0_X$ and $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ and $1\cdot x = x$ and $0\cdot x = 0_X$ and $a\cdot 0_X = 0_X$ and $(-1)\cdot x = -x$ and $x\cdot \mathbf{1}_X = x$ and $\mathbf{1}_X \cdot x = x$ and $x\cdot(y+z) = x\cdot y + x\cdot z$ and $(y+z)\cdot x = y\cdot x + z\cdot x$ and $a\cdot(x\cdot y) = (a\cdot x)\cdot y$ and $a\cdot(x+y) = a\cdot x + a\cdot y$ and $(a+b)\cdot x = a\cdot x + b\cdot x$ and $(a\cdot b)\cdot x = a\cdot(b\cdot x)$ and $(a\cdot b)\cdot(x\cdot y) = a\cdot x\cdot(b\cdot y)$ and $a\cdot(x\cdot y) = x\cdot(a\cdot y)$ and $0_X \cdot x = 0_X$ and $x\cdot 0_X = 0_X$ and $x\cdot(y-z) = x\cdot y - x\cdot z$ and $(y-z)\cdot x = y\cdot x - z\cdot x$ and $(x+y)-z = x+(y-z)$ and $(x-y)+z = x-(y-z)$ and $x-y-z = x-(y+z)$ and $x+y = (x-z)+(z+y)$ and $x-y = (x-z)+(z-y)$ and $x = (x-y)+y$ and $x = y-(y-x)$ and $\|x\| = 0$ iff $x = 0_X$ and $\|a\cdot x\| = |a|\cdot\|x\|$ and $\|x+y\| \leqslant \|x\|+\|y\|$ and $\|x\cdot y\| \leqslant \|x\|\cdot\|y\|$ and $\|\mathbf{1}_X\| = 1$ and X is complete.

Let X be a non empty multiplicative loop structure and let v be an element of X. We say that v is invertible if and only if:

(Def. 8) There exists an element w of X such that $v\cdot w = \mathbf{1}_X$ and $w\cdot v = \mathbf{1}_X$.

Let X be a non empty normed algebra structure, let S be a sequence of X, and let a be an element of X. The functor $a\cdot S$ yielding a sequence of X is defined by:

(Def. 9) For every natural number n holds $(a\cdot S)(n) = a\cdot S(n)$.

Let X be a non empty normed algebra structure, let S be a sequence of X, and let a be an element of X. The functor $S\cdot a$ yields a sequence of X and is defined by:

(Def. 10) For every natural number n holds $(S\cdot a)(n) = S(n)\cdot a$.

Let X be a non empty normed algebra structure and let s_2, s_5 be sequences of X. The functor $s_2\cdot s_5$ yielding a sequence of X is defined as follows:

(Def. 11) For every natural number n holds $(s_2 \cdot s_5)(n) = s_2(n) \cdot s_5(n)$.

Let X be a Banach algebra and let x be an element of X. Let us assume that x is invertible. The functor x^{-1} yielding an element of X is defined as follows:

(Def. 12) $x \cdot x^{-1} = \mathbf{1}_X$ and $x^{-1} \cdot x = \mathbf{1}_X$.

Let X be a Banach algebra and let z be an element of X. The functor $(z^\kappa)_{\kappa\in\mathbb{N}}$ yielding a sequence of X is defined as follows:

(Def. 13) $(z^\kappa)_{\kappa\in\mathbb{N}}(0) = \mathbf{1}_X$ and for every natural number n holds $(z^\kappa)_{\kappa\in\mathbb{N}}(n+1) = (z^\kappa)_{\kappa\in\mathbb{N}}(n) \cdot z$.

Let X be a Banach algebra, let z be an element of X, and let n be a natural number. The functor $z_{\mathbb{N}}^n$ yields an element of X and is defined by:

(Def. 14) $z_{\mathbb{N}}^n = (z^\kappa)_{\kappa\in\mathbb{N}}(n)$.

One can prove the following four propositions:

(44) For every Banach algebra X and for every element z of X holds $z_{\mathbb{N}}^0 = \mathbf{1}_X$.

(45) For every Banach algebra X and for every element z of X such that $\|z\| < 1$ holds $(z^\kappa)_{\kappa\in\mathbb{N}}$ is summable and norm-summable.

(46) Let X be a Banach algebra and x be a point of X. If $\|\mathbf{1}_X - x\| < 1$, then $((\mathbf{1}_X - x)^\kappa)_{\kappa\in\mathbb{N}}$ is summable and $((\mathbf{1}_X - x)^\kappa)_{\kappa\in\mathbb{N}}$ is norm-summable.

(47) For every Banach algebra X and for every point x of X such that $\|\mathbf{1}_X - x\| < 1$ holds x is invertible and $x^{-1} = \sum(((\mathbf{1}_X - x)^\kappa)_{\kappa\in\mathbb{N}})$.

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Formulas and Identities of Trigonometric Functions

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Summary. In this article, we concentrated especially on addition formulas of fundamental trigonometric functions, and their identities.

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The articles [1] and [2] provide the notation and terminology for this paper.

In this paper t_1, t_2, t_3, t_4 denote real numbers.

Let us consider t_1. The functor $\tan t_1$ yielding a real number is defined by:

(Def. 1) $\tan t_1 = \frac{\sin t_1}{\cos t_1}$.

Let us consider t_1. The functor $\cot t_1$ yields a real number and is defined by:

(Def. 2) $\cot t_1 = \frac{\cos t_1}{\sin t_1}$.

Let us consider t_1. The functor $\operatorname{cosec} t_1$ yielding a real number is defined as follows:

(Def. 3) $\operatorname{cosec} t_1 = \frac{1}{\sin t_1}$.

Let us consider t_1. The functor $\sec t_1$ yielding a real number is defined by:

(Def. 4) $\sec t_1 = \frac{1}{\cos t_1}$.

Next we state a number of propositions:

(1) $\tan t_1 = \frac{1}{\cot t_1}$.

(2) $\tan(-t_1) = -\tan t_1$.

(3) $\operatorname{cosec}(-t_1) = -\frac{1}{\sin t_1}$.

(4) $\cot(-t_1) = -\cot t_1$.

(5) If $\cos t_2 \neq 0$, then $\cos t_2 \cdot \sec t_2 = 1$.

(6) $\sin t_1 \cdot \sin t_1 = 1 - \cos t_1 \cdot \cos t_1$.

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(7) $\cos t_1 \cdot \cos t_1 = 1 - \sin t_1 \cdot \sin t_1$.

(8) If $\cos t_1 \neq 0$, then $\sin t_1 = \cos t_1 \cdot \tan t_1$.

(9) $\sin(t_2 - t_3) = \sin t_2 \cdot \cos t_3 - \cos t_2 \cdot \sin t_3$.

(10) $\cos(t_2 - t_3) = \cos t_2 \cdot \cos t_3 + \sin t_2 \cdot \sin t_3$.

(11) If $\cos t_2 \neq 0$ and $\cos t_3 \neq 0$, then $\tan(t_2 + t_3) = \frac{\tan t_2 + \tan t_3}{1 - \tan t_2 \cdot \tan t_3}$.

(12) If $\cos t_2 \neq 0$ and $\cos t_3 \neq 0$, then $\tan(t_2 - t_3) = \frac{\tan t_2 - \tan t_3}{1 + \tan t_2 \cdot \tan t_3}$.

(13) If $\sin t_2 \neq 0$ and $\sin t_3 \neq 0$, then $\cot(t_2 + t_3) = \frac{\cot t_2 \cdot \cot t_3 - 1}{\cot t_3 + \cot t_2}$.

(14) If $\sin t_2 \neq 0$ and $\sin t_3 \neq 0$, then $\cot(t_2 - t_3) = \frac{\cot t_2 \cdot \cot t_3 + 1}{\cot t_3 - \cot t_2}$.

(15) If $\cos t_2 \neq 0$ and $\cos t_3 \neq 0$ and $\cos t_4 \neq 0$, then $\sin(t_2 + t_3 + t_4) = \cos t_2 \cdot \cos t_3 \cdot \cos t_4 \cdot ((\tan t_2 + \tan t_3 + \tan t_4) - \tan t_2 \cdot \tan t_3 \cdot \tan t_4)$.

(16) If $\cos t_2 \neq 0$ and $\cos t_3 \neq 0$ and $\cos t_4 \neq 0$, then $\cos(t_2 + t_3 + t_4) = \cos t_2 \cdot \cos t_3 \cdot \cos t_4 \cdot (1 - \tan t_3 \cdot \tan t_4 - \tan t_4 \cdot \tan t_2 - \tan t_2 \cdot \tan t_3)$.

(17) If $\cos t_2 \neq 0$ and $\cos t_3 \neq 0$ and $\cos t_4 \neq 0$, then $\tan(t_2 + t_3 + t_4) = \frac{(\tan t_2 + \tan t_3 + \tan t_4) - \tan t_2 \cdot \tan t_3 \cdot \tan t_4}{1 - \tan t_3 \cdot \tan t_4 - \tan t_4 \cdot \tan t_2 - \tan t_2 \cdot \tan t_3}$.

(18) If $\sin t_2 \neq 0$ and $\sin t_3 \neq 0$ and $\sin t_4 \neq 0$, then $\cot(t_2 + t_3 + t_4) = \frac{\cot t_2 \cdot \cot t_3 \cdot \cot t_4 - \cot t_2 - \cot t_3 - \cot t_4}{(\cot t_3 \cdot \cot t_4 + \cot t_4 \cdot \cot t_2 + \cot t_2 \cdot \cot t_3) - 1}$.

(19) $\sin t_2 + \sin t_3 = 2 \cdot (\cos(\frac{t_2 - t_3}{2}) \cdot \sin(\frac{t_2 + t_3}{2}))$.

(20) $\sin t_2 - \sin t_3 = 2 \cdot (\cos(\frac{t_2 + t_3}{2}) \cdot \sin(\frac{t_2 - t_3}{2}))$.

(21) $\cos t_2 + \cos t_3 = 2 \cdot (\cos(\frac{t_2 + t_3}{2}) \cdot \cos(\frac{t_2 - t_3}{2}))$.

(22) $\cos t_2 - \cos t_3 = -2 \cdot (\sin(\frac{t_2 + t_3}{2}) \cdot \sin(\frac{t_2 - t_3}{2}))$.

(23) If $\cos t_2 \neq 0$ and $\cos t_3 \neq 0$, then $\tan t_2 + \tan t_3 = \frac{\sin(t_2 + t_3)}{\cos t_2 \cdot \cos t_3}$.

(24) If $\cos t_2 \neq 0$ and $\cos t_3 \neq 0$, then $\tan t_2 - \tan t_3 = \frac{\sin(t_2 - t_3)}{\cos t_2 \cdot \cos t_3}$.

(25) If $\cos t_2 \neq 0$ and $\sin t_3 \neq 0$, then $\tan t_2 + \cot t_3 = \frac{\cos(t_2 - t_3)}{\cos t_2 \cdot \sin t_3}$.

(26) If $\cos t_2 \neq 0$ and $\sin t_3 \neq 0$, then $\tan t_2 - \cot t_3 = -\frac{\cos(t_2 + t_3)}{\cos t_2 \cdot \sin t_3}$.

(27) If $\sin t_2 \neq 0$ and $\sin t_3 \neq 0$, then $\cot t_2 + \cot t_3 = \frac{\sin(t_2 + t_3)}{\sin t_2 \cdot \sin t_3}$.

(28) If $\sin t_2 \neq 0$ and $\sin t_3 \neq 0$, then $\cot t_2 - \cot t_3 = -\frac{\sin(t_2 - t_3)}{\sin t_2 \cdot \sin t_3}$.

(29) $\sin(t_2 + t_3) + \sin(t_2 - t_3) = 2 \cdot (\sin t_2 \cdot \cos t_3)$.

(30) $\sin(t_2 + t_3) - \sin(t_2 - t_3) = 2 \cdot (\cos t_2 \cdot \sin t_3)$.

(31) $\cos(t_2 + t_3) + \cos(t_2 - t_3) = 2 \cdot (\cos t_2 \cdot \cos t_3)$.

(32) $\cos(t_2 + t_3) - \cos(t_2 - t_3) = -2 \cdot (\sin t_2 \cdot \sin t_3)$.

(33) $\sin t_2 \cdot \sin t_3 = -\frac{1}{2} \cdot (\cos(t_2 + t_3) - \cos(t_2 - t_3))$.

(34) $\sin t_2 \cdot \cos t_3 = \frac{1}{2} \cdot (\sin(t_2 + t_3) + \sin(t_2 - t_3))$.

(35) $\cos t_2 \cdot \sin t_3 = \frac{1}{2} \cdot (\sin(t_2 + t_3) - \sin(t_2 - t_3))$.

(36) $\cos t_2 \cdot \cos t_3 = \frac{1}{2} \cdot (\cos(t_2 + t_3) + \cos(t_2 - t_3))$.

(37) $\sin t_2 \cdot \sin t_3 \cdot \sin t_4 = \frac{1}{4} \cdot ((\sin((t_2 + t_3) - t_4) + \sin((t_3 + t_4) - t_2) + \sin((t_4 + t_2) - t_3)) - \sin(t_2 + t_3 + t_4))$.

(38) $\sin t_2 \cdot \sin t_3 \cdot \cos t_4 = \frac{1}{4} \cdot ((-\cos((t_2 + t_3) - t_4) + \cos((t_3 + t_4) - t_2) + \cos((t_4 + t_2) - t_3)) - \cos(t_2 + t_3 + t_4))$.

(39) $\sin t_2 \cdot \cos t_3 \cdot \cos t_4 = \frac{1}{4} \cdot ((\sin((t_2 + t_3) - t_4) - \sin((t_3 + t_4) - t_2)) + \sin((t_4 + t_2) - t_3) + \sin(t_2 + t_3 + t_4))$.

(40) $\cos t_2 \cdot \cos t_3 \cdot \cos t_4 = \frac{1}{4} \cdot (\cos((t_2 + t_3) - t_4) + \cos((t_3 + t_4) - t_2) + \cos((t_4 + t_2) - t_3) + \cos(t_2 + t_3 + t_4))$.

(41) $\sin(t_2 + t_3) \cdot \sin(t_2 - t_3) = \sin t_2 \cdot \sin t_2 - \sin t_3 \cdot \sin t_3$.

(42) $\sin(t_2 + t_3) \cdot \sin(t_2 - t_3) = \cos t_3 \cdot \cos t_3 - \cos t_2 \cdot \cos t_2$.

(43) $\sin(t_2 + t_3) \cdot \cos(t_2 - t_3) = \sin t_2 \cdot \cos t_2 + \sin t_3 \cdot \cos t_3$.

(44) $\cos(t_2 + t_3) \cdot \sin(t_2 - t_3) = \sin t_2 \cdot \cos t_2 - \sin t_3 \cdot \cos t_3$.

(45) $\cos(t_2 + t_3) \cdot \cos(t_2 - t_3) = \cos t_2 \cdot \cos t_2 - \sin t_3 \cdot \sin t_3$.

(46) $\cos(t_2 + t_3) \cdot \cos(t_2 - t_3) = \cos t_3 \cdot \cos t_3 - \sin t_2 \cdot \sin t_2$.

(47) If $\cos t_2 \neq 0$ and $\cos t_3 \neq 0$, then $\frac{\sin(t_2+t_3)}{\sin(t_2-t_3)} = \frac{\tan t_2 + \tan t_3}{\tan t_2 - \tan t_3}$.

(48) If $\cos t_2 \neq 0$ and $\cos t_3 \neq 0$, then $\frac{\cos(t_2+t_3)}{\cos(t_2-t_3)} = \frac{1 - \tan t_2 \cdot \tan t_3}{1 + \tan t_2 \cdot \tan t_3}$.

(49) $\frac{\sin t_2 + \sin t_3}{\sin t_2 - \sin t_3} = \tan(\frac{t_2+t_3}{2}) \cdot \cot(\frac{t_2-t_3}{2})$.

(50) If $\cos(\frac{t_2-t_3}{2}) \neq 0$, then $\frac{\sin t_2 + \sin t_3}{\cos t_2 + \cos t_3} = \tan(\frac{t_2+t_3}{2})$.

(51) If $\cos(\frac{t_2+t_3}{2}) \neq 0$, then $\frac{\sin t_2 - \sin t_3}{\cos t_2 + \cos t_3} = \tan(\frac{t_2-t_3}{2})$.

(52) If $\sin(\frac{t_2+t_3}{2}) \neq 0$, then $\frac{\sin t_2 + \sin t_3}{\cos t_3 - \cos t_2} = \cot(\frac{t_2-t_3}{2})$.

(53) If $\sin(\frac{t_2-t_3}{2}) \neq 0$, then $\frac{\sin t_2 - \sin t_3}{\cos t_3 - \cos t_2} = \cot(\frac{t_2+t_3}{2})$.

(54) $\frac{\cos t_2 + \cos t_3}{\cos t_2 - \cos t_3} = \cot(\frac{t_2+t_3}{2}) \cdot \cot(\frac{t_3-t_2}{2})$.

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The Class of Series-Parallel Graphs. Part III

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Summary. This paper contains some facts and theorems relating to the following operations on graphs: union, sum, complement and "embeds". We also introduce connected graphs to prove that a finite irreflexive symmetric N-free graph is a finite series-parallel graph. This article continues the formalization of [22].

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The papers [25], [24], [28], [12], [29], [31], [30], [2], [13], [1], [27], [18], [17], [8], [14], [16], [20], [23], [7], [10], [26], [11], [4], [6], [19], [15], [5], [21], [3], and [9] provide the notation and terminology for this paper.

1. Preliminaries

In this paper A, B, a, b, c, d, e, f, g, h denote sets.

One can prove the following three propositions:

(1) $\mathrm{id}_A \restriction B = \mathrm{id}_A \cap [\!: B, B :\!]$.

(2) $\mathrm{id}_{\{a,b,c,d\}} = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$.

(3) $[\!: \{a,b,c,d\}, \{e,f,g,h\} :\!] = \{\langle a, e\rangle, \langle a, f\rangle, \langle b, e\rangle, \langle b, f\rangle, \langle a, g\rangle, \langle a, h\rangle, \langle b, g\rangle, \langle b, h\rangle\} \cup \{\langle c, e\rangle, \langle c, f\rangle, \langle d, e\rangle, \langle d, f\rangle, \langle c, g\rangle, \langle c, h\rangle, \langle d, g\rangle, \langle d, h\rangle\}$.

Let X, Y be trivial sets. Observe that every relation between X and Y is trivial.

We now state the proposition

(4) For every trivial set X and for every binary relation R on X such that R is non empty there exists a set x such that $R = \{\langle x, x\rangle\}$.

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Let X be a trivial set. Observe that every binary relation on X is trivial, reflexive, symmetric, transitive, and strongly connected.

We now state the proposition

(5) For every non empty trivial set X holds every binary relation on X is symmetric in X.

One can verify that there exists a relational structure which is non empty, strict, finite, irreflexive, and symmetric.

Let L be an irreflexive relational structure. Observe that every full relational substructure of L is irreflexive.

Let L be a symmetric relational structure. Note that every full relational substructure of L is symmetric.

One can prove the following proposition

(6) Let R be an irreflexive symmetric relational structure. Suppose $\overline{\overline{\text{the carrier of } R}} = 2$. Then there exist sets a, b such that the carrier of $R = \{a, b\}$ but the internal relation of $R = \{\langle a, b\rangle, \langle b, a\rangle\}$ or the internal relation of $R = \emptyset$.

2. Some Facts about Operations "UnionOf" and "SumOf"

Let R be a non empty relational structure and let S be a relational structure. Note that $\mathrm{UnionOf}(R, S)$ is non empty and $\mathrm{SumOf}(R, S)$ is non empty.

Let R be a relational structure and let S be a non empty relational structure. Observe that $\mathrm{UnionOf}(R, S)$ is non empty and $\mathrm{SumOf}(R, S)$ is non empty.

Let R, S be finite relational structures. One can check that $\mathrm{UnionOf}(R, S)$ is finite and $\mathrm{SumOf}(R, S)$ is finite.

Let R, S be symmetric relational structures. One can check that $\mathrm{UnionOf}(R, S)$ is symmetric and $\mathrm{SumOf}(R, S)$ is symmetric.

Let R, S be irreflexive relational structures. Observe that $\mathrm{UnionOf}(R, S)$ is irreflexive.

The following four propositions are true:

(7) Let R, S be irreflexive relational structures. Suppose the carrier of R misses the carrier of S. Then $\mathrm{SumOf}(R, S)$ is irreflexive.

(8) For all relational structures R_1, R_2 holds $\mathrm{UnionOf}(R_1, R_2) = \mathrm{UnionOf}(R_2, R_1)$ and $\mathrm{SumOf}(R_1, R_2) = \mathrm{SumOf}(R_2, R_1)$.

(9) Let G be an irreflexive relational structure and G_1, G_2 be relational structures. If $G = \mathrm{UnionOf}(G_1, G_2)$ or $G = \mathrm{SumOf}(G_1, G_2)$, then G_1 is irreflexive and G_2 is irreflexive.

(10) Let G be a non empty relational structure and H_1, H_2 be relational structures. Suppose that

(i) the carrier of H_1 misses the carrier of H_2, and

(ii) the relational structure of $G = \mathrm{UnionOf}(H_1, H_2)$ or the relational structure of $G = \mathrm{SumOf}(H_1, H_2)$.
Then H_1 is a full relational substructure of G and H_2 is a full relational substructure of G.

3. Theorems Relating to the Complement of Relational Structure

One can prove the following proposition

(11) The internal relation of ComplRelStr Necklace 4 $= \{\langle 0, 2\rangle, \langle 2, 0\rangle, \langle 0, 3\rangle, \langle 3, 0\rangle, \langle 1, 3\rangle, \langle 3, 1\rangle\}$.

Let R be a relational structure. Note that ComplRelStr R is irreflexive.

Let R be a symmetric relational structure. Note that ComplRelStr R is symmetric.

Next we state several propositions:

(12) For every relational structure R holds the internal relation of R misses the internal relation of ComplRelStr R.

(13) For every relational structure R holds $\mathrm{id}_{\text{the carrier of } R}$ misses the internal relation of ComplRelStr R.

(14) Let G be a relational structure. Then $[\!:$ the carrier of G, the carrier of $G\,:\!] = \mathrm{id}_{\text{the carrier of } G} \cup$ the internal relation of $G \cup$ the internal relation of ComplRelStr G.

(15) For every strict irreflexive relational structure G such that G is trivial holds ComplRelStr $G = G$.

(16) For every strict irreflexive relational structure G holds ComplRelStr ComplRelStr $G = G$.

(17) For all relational structures G_1, G_2 such that the carrier of G_1 misses the carrier of G_2 holds ComplRelStr $\mathrm{UnionOf}(G_1, G_2) = \mathrm{SumOf}(\mathrm{ComplRelStr}\, G_1, \mathrm{ComplRelStr}\, G_2)$.

(18) For all relational structures G_1, G_2 such that the carrier of G_1 misses the carrier of G_2 holds ComplRelStr $\mathrm{SumOf}(G_1, G_2) = \mathrm{UnionOf}(\mathrm{ComplRelStr}\, G_1, \mathrm{ComplRelStr}\, G_2)$.

(19) Let G be a relational structure and H be a full relational substructure of G. Then the internal relation of ComplRelStr H = (the internal relation of ComplRelStr G) $|^2$ (the carrier of ComplRelStr H).

(20) Let G be a non empty irreflexive relational structure, x be an element of the carrier of G, and x' be an element of the carrier of ComplRelStr G. If $x = x'$, then $\mathrm{ComplRelStr}\,\mathrm{sub}(\Omega_G \setminus \{x\}) = \mathrm{sub}(\Omega_{\mathrm{ComplRelStr}\, G} \setminus \{x'\})$.

4. Another Facts Relating to Operation "embeds"

Let us observe that every non empty relational structure which is trivial and strict is also N-free.

The following propositions are true:

(21) Let R be a reflexive antisymmetric relational structure and S be a relational structure. Then there exists a map f from R into S such that for all elements x, y of the carrier of R holds $\langle x, y\rangle \in$ the internal relation of R iff $\langle f(x), f(y)\rangle \in$ the internal relation of S if and only if S embeds R.

(22) Let G be a non empty relational structure and H be a non empty full relational substructure of G. Then G embeds H.

(23) Let G be a non empty relational structure and H be a non empty full relational substructure of G. If G is N-free, then H is N-free.

(24) For every non empty irreflexive relational structure G holds G embeds Necklace 4 iff ComplRelStr G embeds Necklace 4.

(25) For every non empty irreflexive relational structure G holds G is N-free iff ComplRelStr G is N-free.

5. Connected Graphs

Let R be a relational structure. A path of R is a reduction sequence w.r.t. the internal relation of R.

Let R be a relational structure. We say that R is path-connected if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let x, y be sets. Suppose $x \in$ the carrier of R and $y \in$ the carrier of R and $x \neq y$. Then the internal relation of R reduces x to y or the internal relation of R reduces y to x.

One can check that every relational structure which is empty is also path-connected.

One can check that every non empty relational structure which is connected is also path-connected.

We now state the proposition

(26) Let R be a non empty transitive reflexive relational structure and x, y be elements of R. Suppose the internal relation of R reduces x to y. Then $\langle x, y\rangle \in$ the internal relation of R.

One can check that every non empty transitive reflexive relational structure which is path-connected is also connected.

Next we state the proposition

(27) Let R be a symmetric relational structure and x, y be sets. Suppose $x \in$ the carrier of R and $y \in$ the carrier of R. Suppose the internal relation of R reduces x to y. Then the internal relation of R reduces y to x.

Let R be a symmetric relational structure. Let us observe that R is path-connected if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let x, y be sets. Suppose $x \in$ the carrier of R and $y \in$ the carrier of R and $x \neq y$. Then the internal relation of R reduces x to y.

Let R be a relational structure and let x be an element of R. The functor component(x) yielding a subset of R is defined as follows:

(Def. 3) $\mathrm{component}(x) = [x]_{\mathrm{EqCl}(\text{the internal relation of } R)}$.

Next we state the proposition

(28) For every non empty relational structure R and for every element x of R holds $x \in \mathrm{component}(x)$.

Let R be a non empty relational structure and let x be an element of R. Note that component(x) is non empty.

Next we state a number of propositions:

(29) Let R be a relational structure, x be an element of R, and y be a set. If $y \in \mathrm{component}(x)$, then $\langle x, y\rangle \in$ EqCl(the internal relation of R).

(30) Let R be a relational structure, x be an element of R, and A be a set. Then $A = \mathrm{component}(x)$ if and only if for every set y holds $y \in A$ iff $\langle x, y\rangle \in$ EqCl(the internal relation of R).

(31) Let R be a non empty irreflexive symmetric relational structure. Suppose R is not path-connected. Then there exist non empty strict irreflexive symmetric relational structures G_1, G_2 such that the carrier of G_1 misses the carrier of G_2 and the relational structure of $R = \mathrm{UnionOf}(G_1, G_2)$.

(32) Let R be a non empty irreflexive symmetric relational structure. Suppose ComplRelStr R is not path-connected. Then there exist non empty strict irreflexive symmetric relational structures G_1, G_2 such that the carrier of G_1 misses the carrier of G_2 and the relational structure of $R = \mathrm{SumOf}(G_1, G_2)$.

(33) For every irreflexive relational structure G such that $G \in$ FinRelStrSp holds ComplRelStr $G \in$ FinRelStrSp .

(34) Let R be an irreflexive symmetric relational structure. Suppose $\overline{\overline{\text{the carrier of } R}} = 2$ and the carrier of $R \in \mathbf{U}_0$. Then the relational structure of $R \in$ FinRelStrSp .

(35) For every relational structure R such that $R \in$ FinRelStrSp holds R is symmetric.

(36) Let G be a relational structure, H_1, H_2 be non empty relational structures, x be an element of the carrier of H_1, and y be an element of the

carrier of H_2. Suppose $G = \mathrm{UnionOf}(H_1, H_2)$ and the carrier of H_1 misses the carrier of H_2. Then $\langle x, y \rangle \notin$ the internal relation of G.

(37) Let G be a relational structure, H_1, H_2 be non empty relational structures, x be an element of the carrier of H_1, and y be an element of the carrier of H_2. If $G = \mathrm{SumOf}(H_1, H_2)$, then $\langle x, y \rangle \notin$ the internal relation of ComplRelStr G.

(38) Let G be a non empty symmetric relational structure, x be an element of the carrier of G, and R_1, R_2 be non empty relational structures. Suppose the carrier of R_1 misses the carrier of R_2 and $\mathrm{sub}(\Omega_G \setminus \{x\}) = \mathrm{UnionOf}(R_1, R_2)$ and G is path-connected. Then there exists an element b of the carrier of R_1 such that $\langle b, x \rangle \in$ the internal relation of G.

(39) Let G be a non empty symmetric irreflexive relational structure, a, b, c, d be elements of the carrier of G, and Z be a subset of the carrier of G. Suppose that $Z = \{a, b, c, d\}$ and a, b, c, d are mutually different and $\langle a, b \rangle \in$ the internal relation of G and $\langle b, c \rangle \in$ the internal relation of G and $\langle c, d \rangle \in$ the internal relation of G and $\langle a, c \rangle \notin$ the internal relation of G and $\langle a, d \rangle \notin$ the internal relation of G and $\langle b, d \rangle \notin$ the internal relation of G. Then $\mathrm{sub}(Z)$ embeds Necklace 4.

(40) Let G be a non empty irreflexive symmetric relational structure, x be an element of the carrier of G, and R_1, R_2 be non empty relational structures. Suppose that

(i) the carrier of R_1 misses the carrier of R_2,
(ii) $\mathrm{sub}(\Omega_G \setminus \{x\}) = \mathrm{UnionOf}(R_1, R_2)$,
(iii) G is non trivial and path-connected, and
(iv) ComplRelStr G is path-connected.

Then G embeds Necklace 4.

(41) Let G be a non empty strict finite irreflexive symmetric relational structure. Suppose G is N-free and the carrier of $G \in \mathbf{U}_0$. Then the relational structure of $G \in$ FinRelStrSp .

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Relocability for SCM over Ring

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The notation and terminology used in this paper have been introduced in the following articles: [23], [27], [3], [4], [10], [28], [21], [7], [8], [5], [22], [1], [26], [6], [9], [19], [2], [15], [18], [16], [17], [24], [20], [12], [11], [25], [13], and [14].

1. On the Standard Computers

For simplicity, we use the following convention: i, j, k denote natural numbers, n denotes a natural number, N denotes a set with non empty elements, S denotes a standard IC-Ins-separated definite non empty non void AMI over N, l denotes an instruction-location of S, and f denotes a finite partial state of S.

Next we state the proposition

(1) $\mathbb{N} \approx$ the instruction locations of S.

Let us consider N, S. Observe that the instruction locations of S is infinite.

We now state the proposition

(2) $\mathrm{il}_S(i) + j = \mathrm{il}_S(i + j)$.

Let N be a set with non empty elements, let S be a standard IC-Ins-separated definite non empty non void AMI over N, let l_1 be an instruction-location of S, and let k be a natural number. The functor $l_1 -' k$ yields an instruction-location of S and is defined as follows:

(Def. 1) $l_1 -' k = \mathrm{il}_S(\mathrm{locnum}(l_1) -' k)$.

We now state a number of propositions:

(3) $l -' 0 = l$.

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(4) $\mathrm{locnum}(l) -' k = \mathrm{locnum}(l -' k)$.

(5) $(l + k) -' k = l$.

(6) $\mathrm{il}_S(i) -' j = \mathrm{il}_S(i -' j)$.

(7) Let S be an IC-Ins-separated definite non empty non void AMI over N and p be a finite partial state of S. Then $\mathrm{dom}\,\mathrm{DataPart}(p) \subseteq$ (the carrier of S) $\setminus$ ($\{\mathbf{IC}_S\} \cup$ the instruction locations of S).

(8) Let S be an IC-Ins-separated definite realistic non empty non void AMI over N and p be a finite partial state of S. Then p is data-only if and only if $\mathrm{dom}\, p \subseteq$ (the carrier of S) $\setminus$ ($\{\mathbf{IC}_S\} \cup$ the instruction locations of S).

(9) For all instruction-locations l_2, l_3 of S holds $\text{Start-At}(l_2 + k) = \text{Start-At}(l_3 + k)$ iff $\text{Start-At}(l_2) = \text{Start-At}(l_3)$.

(10) For all instruction-locations l_2, l_3 of S such that $\text{Start-At}(l_2) = \text{Start-At}(l_3)$ holds $\text{Start-At}(l_2 -' k) = \text{Start-At}(l_3 -' k)$.

(11) If $l \in \mathrm{dom}\, f$, then $(\mathrm{Shift}(f, k))(l + k) = f(l)$.

(12) $\mathrm{dom}\,\mathrm{Shift}(f, k) = \{i_1 + k; i_1$ ranges over instruction-locations of S: $i_1 \in \mathrm{dom}\, f\}$.

(13) Let S be an Exec-preserving IC-Ins-separated definite realistic steady-programmed non empty non void AMI over N, s be a state of S, i be an instruction of S, and p be a programmed finite partial state of S. Then $\mathrm{Exec}(i, s{+\cdot}p) = \mathrm{Exec}(i, s){+\cdot}p$.

2. $\mathbf{SCM}(R)$

For simplicity, we follow the rules: R denotes a good ring, a, b denote Data-Locations of R, l_1 denotes an instruction-location of $\mathbf{SCM}(R)$, I denotes an instruction of $\mathbf{SCM}(R)$, p denotes a finite partial state of $\mathbf{SCM}(R)$, s, s_1, s_2 denote states of $\mathbf{SCM}(R)$, and q denotes a finite partial state of $\mathbf{SCM}$.

One can prove the following propositions:

(14) The carrier of $\mathbf{SCM}(R) = \{\mathbf{IC}_{\mathbf{SCM}(R)}\} \cup \text{Data-Loc}_{\mathrm{SCM}} \cup \text{Instr-Loc}_{\mathrm{SCM}}$.

(15) $\mathrm{ObjectKind}(l_1) = \mathrm{Instr}_{\mathrm{SCM}}(R)$.

(16) $\mathrm{dl}_R(n) = 2 \cdot n + 1$.

(17) $\mathrm{il}_{\mathbf{SCM}(R)}(k) = 2 \cdot k + 2$.

(18) For every Data-Location d_1 of R there exists a natural number i such that $d_1 = \mathrm{dl}_R(i)$.

(19) For all natural numbers i, j such that $i \neq j$ holds $\mathrm{dl}_R(i) \neq \mathrm{dl}_R(j)$.

(20) $a \neq l_1$.

(21) $\text{Data-Loc}_{\mathrm{SCM}} \subseteq \mathrm{dom}\, s$.

(22) $\mathrm{dom}(s{\restriction}\text{Data-Loc}_{\mathrm{SCM}}) = \text{Data-Loc}_{\mathrm{SCM}}$.

(23) If $p = q$, then $\mathrm{DataPart}(p) = \mathrm{DataPart}(q)$.

(24) $\mathrm{DataPart}(p) = p \restriction \text{Data-Loc}_{\mathrm{SCM}}$.

(25) p is data-only iff $\mathrm{dom}\, p \subseteq \text{Data-Loc}_{\mathrm{SCM}}$.

(26) $\mathrm{dom}\, \mathrm{DataPart}(p) \subseteq \text{Data-Loc}_{\mathrm{SCM}}$.

(27) $\text{Instr-Loc}_{\mathrm{SCM}} \subseteq \mathrm{dom}\, s$.

(28) If $p = q$, then $\mathrm{ProgramPart}(p) = \mathrm{ProgramPart}(q)$.

(29) $\mathrm{dom}\, \mathrm{ProgramPart}(p) \subseteq \text{Instr-Loc}_{\mathrm{SCM}}$.

Let us consider R and let I be an element of the instructions of $\mathbf{SCM}(R)$. Observe that $\mathrm{InsCode}(I)$ is natural.

Next we state several propositions:

(30) $\mathrm{InsCode}(I) \leqslant 7$.

(31) $\mathrm{IncAddr}(\mathrm{goto}\ l_1, k) = \mathrm{goto}\ (l_1 + k)$.

(32) $\mathrm{IncAddr}(\mathbf{if}\ a = 0\ \mathbf{goto}\ l_1, k) = \mathbf{if}\ a = 0\ \mathbf{goto}\ l_1 + k$.

(33) $s(a) = (s {+\cdot}\, \text{Start-At}(l_1))(a)$.

(34) Suppose $\mathbf{IC}_{(s_1)} = \mathbf{IC}_{(s_2)}$ and for every Data-Location a of R holds $s_1(a) = s_2(a)$ and for every instruction-location i of $\mathbf{SCM}(R)$ holds $s_1(i) = s_2(i)$. Then $s_1 = s_2$.

(35) $\mathrm{Exec}(\mathrm{IncAddr}(\mathrm{CurInstr}(s), k), s {+\cdot}\, \text{Start-At}(\mathbf{IC}_s + k)) = \mathrm{Following}(s) {+\cdot}\, \text{Start-At}(\mathbf{IC}_{\mathrm{Following}(s)} + k)$.

(36) If $\mathbf{IC}_s = \mathrm{il}_{\mathbf{SCM}(R)}(j + k)$, then $\mathrm{Exec}(I, s {+\cdot}\, \text{Start-At}(\mathbf{IC}_s -' k)) = \mathrm{Exec}(\mathrm{IncAddr}(I, k), s) {+\cdot}\, \text{Start-At}(\mathbf{IC}_{\mathrm{Exec}(\mathrm{IncAddr}(I,k),s)} -' k)$.

Let us consider R. One can check that there exists a finite partial state of $\mathbf{SCM}(R)$ which is autonomic and non programmed.

Let us consider R, let a be a Data-Location of R, and let r be an element of the carrier of R. Then $a \mapsto r$ is a finite partial state of $\mathbf{SCM}(R)$.

We now state a number of propositions:

(37) If R is non trivial, then for every autonomic finite partial state p of $\mathbf{SCM}(R)$ such that $\mathrm{DataPart}(p) \neq \emptyset$ holds $\mathbf{IC}_{\mathbf{SCM}(R)} \in \mathrm{dom}\, p$.

(38) If R is non trivial, then for every autonomic non programmed finite partial state p of $\mathbf{SCM}(R)$ holds $\mathbf{IC}_{\mathbf{SCM}(R)} \in \mathrm{dom}\, p$.

(39) For every autonomic finite partial state p of $\mathbf{SCM}(R)$ such that $\mathbf{IC}_{\mathbf{SCM}(R)} \in \mathrm{dom}\, p$ holds $\mathbf{IC}_p \in \mathrm{dom}\, p$.

(40) Suppose R is non trivial. Let p be an autonomic non programmed finite partial state of $\mathbf{SCM}(R)$. If $p \subseteq s$, then $\mathbf{IC}_{(\mathrm{Computation}(s))(n)} \in \mathrm{dom}\, \mathrm{ProgramPart}(p)$.

(41) Suppose R is non trivial. Let p be an autonomic non programmed finite partial state of $\mathbf{SCM}(R)$. If $p \subseteq s_1$ and $p \subseteq s_2$, then $\mathbf{IC}_{(\mathrm{Computation}(s_1))(n)} = \mathbf{IC}_{(\mathrm{Computation}(s_2))(n)}$ and $\mathrm{CurInstr}((\mathrm{Computation}(s_1))(n)) = \mathrm{CurInstr}((\mathrm{Computation}(s_2))(n))$.

(42) Suppose R is non trivial. Let p be an autonomic non programmed finite partial state of $\mathbf{SCM}(R)$. If $p \subseteq s_1$ and $p \subseteq s_2$ and $\mathrm{CurInstr}((\mathrm{Computation}(s_1))(n)) = a{:=}b$ and $a \in \mathrm{dom}\, p$, then $(\mathrm{Computation}(s_1))(n)(b) = (\mathrm{Computation}(s_2))(n)(b)$.

(43) Suppose R is non trivial. Let p be an autonomic non programmed finite partial state of $\mathbf{SCM}(R)$. Suppose $p \subseteq s_1$ and $p \subseteq s_2$ and $\mathrm{CurInstr}((\mathrm{Computation}(s_1))(n)) = \mathrm{AddTo}(a, b)$ and $a \in \mathrm{dom}\, p$. Then $(\mathrm{Computation}(s_1))(n)(a) + (\mathrm{Computation}(s_1))(n)(b) = (\mathrm{Computation}(s_2))(n)(a) + (\mathrm{Computation}(s_2))(n)(b)$.

(44) Suppose R is non trivial. Let p be an autonomic non programmed finite partial state of $\mathbf{SCM}(R)$. Suppose $p \subseteq s_1$ and $p \subseteq s_2$ and $\mathrm{CurInstr}((\mathrm{Computation}(s_1))(n)) = \mathrm{SubFrom}(a, b)$ and $a \in \mathrm{dom}\, p$. Then $(\mathrm{Computation}(s_1))(n)(a) - (\mathrm{Computation}(s_1))(n)(b) = (\mathrm{Computation}(s_2))(n)(a) - (\mathrm{Computation}(s_2))(n)(b)$.

(45) Suppose R is non trivial. Let p be an autonomic non programmed finite partial state of $\mathbf{SCM}(R)$. Suppose $p \subseteq s_1$ and $p \subseteq s_2$ and $\mathrm{CurInstr}((\mathrm{Computation}(s_1))(n)) = \mathrm{MultBy}(a, b)$ and $a \in \mathrm{dom}\, p$. Then $(\mathrm{Computation}(s_1))(n)(a) \cdot (\mathrm{Computation}(s_1))(n)(b) = (\mathrm{Computation}(s_2))(n)(a) \cdot (\mathrm{Computation}(s_2))(n)(b)$.

(46) Suppose R is non trivial. Let p be an autonomic non programmed finite partial state of $\mathbf{SCM}(R)$. Suppose $p \subseteq s_1$ and $p \subseteq s_2$ and $\mathrm{CurInstr}((\mathrm{Computation}(s_1))(n)) = \mathbf{if}\ a = 0\ \mathbf{goto}\ l_1$ and $l_1 \neq \mathrm{Next}(\mathbf{IC}_{(\mathrm{Computation}(s_1))(n)})$. Then $(\mathrm{Computation}(s_1))(n)(a) = 0_R$ if and only if $(\mathrm{Computation}(s_2))(n)(a) = 0_R$.

3. Relocability

Let N be a set with non empty elements, let S be a regular standard IC-Ins-separated definite non empty non void AMI over N, let k be a natural number, and let p be a finite partial state of S. The functor $\mathrm{Relocated}(p, k)$ yielding a finite partial state of S is defined as follows:

(Def. 2) $\mathrm{Relocated}(p, k) = \text{Start-At}(\mathbf{IC}_p + k) {+\cdot}\, \mathrm{IncAddr}(\mathrm{Shift}(\mathrm{ProgramPart}(p), k), k) {+\cdot}\, \mathrm{DataPart}(p)$.

In the sequel S denotes a regular standard IC-Ins-separated definite non empty non void AMI over N, g denotes a finite partial state of S, and i_1 denotes an instruction-location of S.

One can prove the following propositions:

(47) $\mathrm{DataPart}(\mathrm{Relocated}(g, k)) = \mathrm{DataPart}(g)$.

(48) If S is realistic, then $\mathrm{ProgramPart}(\mathrm{Relocated}(g, k)) = \mathrm{IncAddr}(\mathrm{Shift}(\mathrm{ProgramPart}(g), k), k)$.

(49) If S is realistic, then dom ProgramPart(Relocated(g, k)) $= \{\text{il}_S(j+k); j$ ranges over natural numbers: $\text{il}_S(j) \in$ dom ProgramPart$(g)\}$.

(50) If S is realistic, then $i_1 \in \text{dom}\, g$ iff $i_1 + k \in$ dom Relocated(g, k).

(51) $\mathbf{IC}_S \in$ dom Relocated(g, k).

(52) If S is realistic, then $\mathbf{IC}_{\text{Relocated}(g,k)} = \mathbf{IC}_g + k$.

(53) Let p be a programmed finite partial state of S and l be an instruction-location of S. If $l \in \text{dom}\, p$, then (IncAddr(p, k))(l) = IncAddr$(\pi_l p, k)$.

(54) For every programmed finite partial state p of S holds Shift(IncAddr$(p, i), i)$ = IncAddr(Shift$(p, i), i)$.

(55) If S is realistic, then for every instruction I of S such that $i_1 \in$ dom ProgramPart(g) and $I = g(i_1)$ holds IncAddr(I, k) = (Relocated(g, k))$(i_1 + k)$.

(56) If S is realistic, then Start-At$(\mathbf{IC}_g + k) \subseteq$ Relocated(g, k).

(57) If S is realistic, then for every data-only finite partial state q of S such that $\mathbf{IC}_S \in \text{dom}\, g$ holds Relocated$(g{+\cdot}q, k)$ = Relocated$(g, k){+\cdot}q$.

(58) For every autonomic finite partial state p of $\mathbf{SCM}(R)$ such that $p \subseteq s_1$ and Relocated$(p, k) \subseteq s_2$ holds $p \subseteq s_1{+\cdot}s_2{\restriction}\text{Data-Loc}_{\text{SCM}}$.

(59) Suppose R is non trivial. Let p be an autonomic finite partial state of $\mathbf{SCM}(R)$. Suppose $\mathbf{IC}_{\mathbf{SCM}(R)} \in \text{dom}\, p$ and $p \subseteq s_1$ and Relocated$(p, k) \subseteq s_2$ and $s = s_1{+\cdot}s_2{\restriction}\text{Data-Loc}_{\text{SCM}}$. Let i be a natural number. Then $\mathbf{IC}_{(\text{Computation}(s_1))(i)} + k = \mathbf{IC}_{(\text{Computation}(s_2))(i)}$ and IncAddr(CurInstr((Computation(s_1))(i))$, k)$ = CurInstr((Computation(s_2))(i)) and (Computation(s_1))$(i){\restriction}$dom DataPart (p) = (Computation(s_2))$(i){\restriction}$dom DataPart(Relocated(p, k)) and (Computation(s))$(i){\restriction}\text{Data-Loc}_{\text{SCM}}$ = (Computation(s_2))$(i){\restriction}\text{Data-Loc}_{\text{SCM}}$.

(60) Suppose R is non trivial. Let p be an autonomic finite partial state of $\mathbf{SCM}(R)$. If $\mathbf{IC}_{\mathbf{SCM}(R)} \in \text{dom}\, p$, then p is halting iff Relocated(p, k) is halting.

(61) Suppose R is non trivial. Let p be an autonomic finite partial state of $\mathbf{SCM}(R)$. Suppose $\mathbf{IC}_{\mathbf{SCM}(R)} \in \text{dom}\, p$ and $p \subseteq s$. Let i be a natural number. Then (Computation$(s{+\cdot}$ Relocated(p, k)))(i) = (Computation(s))$(i){+\cdot}$ Start-At$(\mathbf{IC}_{(\text{Computation}(s))(i)} + k){+\cdot}$ ProgramPart (Relocated(p, k)).

(62) Suppose R is non trivial. Let p be an autonomic finite partial state of $\mathbf{SCM}(R)$. Suppose $\mathbf{IC}_{\mathbf{SCM}(R)} \in \text{dom}\, p$ and Relocated$(p, k) \subseteq s$. Let i be a natural number. Then (Computation(s))(i) = (Computation$(s{+\cdot}p)$)$(i){+\cdot}$ Start-At$(\mathbf{IC}_{(\text{Computation}(s{+\cdot}p))(i)} + k){+\cdot}s{\restriction}$dom ProgramPart$(p){+\cdot}$ ProgramPart(Relocated(p, k)).

(63) Suppose R is non trivial and $\mathbf{IC}_{\mathbf{SCM}(R)} \in \text{dom}\, p$ and $p \subseteq s$ and Relocated(p, k) is autonomic. Let i be a natural number. Then

(Computation(s))(i) = (Computation(s+· Relocated(p,k)))(i)+· Start-At ($\mathbf{IC}_{(\mathrm{Computation}(s+\cdot\,\mathrm{Relocated}(p,k)))(i)} -' k$)+·$s\restriction$ dom ProgramPart(Relocated(p, k))+· ProgramPart(p).

(64) If R is non trivial and $\mathbf{IC}_{\mathbf{SCM}(R)} \in \operatorname{dom} p$, then p is autonomic iff Relocated(p,k) is autonomic.

(65) Suppose R is non trivial. Let p be a halting autonomic finite partial state of $\mathbf{SCM}(R)$. If $\mathbf{IC}_{\mathbf{SCM}(R)} \in \operatorname{dom} p$, then DataPart(Result($p$)) = DataPart(Result(Relocated(p,k))).

(66) Suppose R is non trivial. Let F be a partial function from FinPartSt($\mathbf{SCM}(R)$) to FinPartSt($\mathbf{SCM}(R)$). Suppose $\mathbf{IC}_{\mathbf{SCM}(R)} \in \operatorname{dom} p$ and F is data-only. Then p computes F if and only if Relocated(p,k) computes F.

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Convergent Sequences in Complex Unitary Space

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Summary. In this article, we introduce the notion of convergence sequence in complex unitary space and complex Hilbert space.

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The terminology and notation used in this paper are introduced in the following papers: [15], [2], [14], [7], [1], [17], [3], [4], [10], [9], [16], [13], [11], [12], [8], [5], and [6].

1. Convergence in Complex Unitary Space

For simplicity, we adopt the following convention: X is a complex unitary space, x, y, w, g, g_1, g_2 are points of X, z is a Complex, q, r, M are real numbers, s_1, s_2, s_3, s_4 are sequences of X, k, n, m are natural numbers, and N_1 is an increasing sequence of naturals.

Let us consider X, s_1. We say that s_1 is convergent if and only if:

(Def. 1) There exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\rho(s_1(n), g) < r$.

Next we state several propositions:

(1) If s_1 is constant, then s_1 is convergent.

(2) If s_2 is convergent and there exists k such that for every n such that $k \leqslant n$ holds $s_3(n) = s_2(n)$, then s_3 is convergent.

(3) If s_2 is convergent and s_3 is convergent, then $s_2 + s_3$ is convergent.

(4) If s_2 is convergent and s_3 is convergent, then $s_2 - s_3$ is convergent.

(5) If s_1 is convergent, then $z \cdot s_1$ is convergent.

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(6) If s_1 is convergent, then $-s_1$ is convergent.

(7) If s_1 is convergent, then $s_1 + x$ is convergent.

(8) If s_1 is convergent, then $s_1 - x$ is convergent.

(9) s_1 is convergent if and only if there exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields a point of X and is defined as follows:

(Def. 2) For every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\rho(s_1(n), \lim s_1) < r$.

One can prove the following propositions:

(10) If s_1 is constant and $x \in \operatorname{rng} s_1$, then $\lim s_1 = x$.

(11) If s_1 is constant and there exists n such that $s_1(n) = x$, then $\lim s_1 = x$.

(12) If s_2 is convergent and there exists k such that for every n such that $n \geqslant k$ holds $s_3(n) = s_2(n)$, then $\lim s_2 = \lim s_3$.

(13) If s_2 is convergent and s_3 is convergent, then $\lim(s_2+s_3) = \lim s_2+\lim s_3$.

(14) If s_2 is convergent and s_3 is convergent, then $\lim(s_2-s_3) = \lim s_2-\lim s_3$.

(15) If s_1 is convergent, then $\lim(z \cdot s_1) = z \cdot \lim s_1$.

(16) If s_1 is convergent, then $\lim(-s_1) = -\lim s_1$.

(17) If s_1 is convergent, then $\lim(s_1 + x) = \lim s_1 + x$.

(18) If s_1 is convergent, then $\lim(s_1 - x) = \lim s_1 - x$.

(19) Suppose s_1 is convergent. Then $\lim s_1 = g$ if and only if for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. The functor $\|s_1\|$ yielding a sequence of real numbers is defined as follows:

(Def. 3) For every n holds $\|s_1\|(n) = \|s_1(n)\|$.

One can prove the following three propositions:

(20) If s_1 is convergent, then $\|s_1\|$ is convergent.

(21) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1\|$ is convergent and $\lim\|s_1\| = \|g\|$.

(22) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - g\|$ is convergent and $\lim\|s_1 - g\| = 0$.

Let us consider X, s_1, x. The functor $\rho(s_1, x)$ yielding a sequence of real numbers is defined as follows:

(Def. 4) For every n holds $(\rho(s_1, x))(n) = \rho(s_1(n), x)$.

One can prove the following propositions:

(23) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent.

(24) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent and $\lim \rho(s_1, g) = 0$.

(25) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 + s_3\|$ is convergent and $\lim\|s_2 + s_3\| = \|g_1 + g_2\|$.

(26) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|(s_2+s_3)-(g_1+g_2)\|$ is convergent and $\lim\|(s_2+s_3)-(g_1+g_2)\| = 0$.

(27) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3\|$ is convergent and $\lim\|s_2 - s_3\| = \|g_1 - g_2\|$.

(28) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3 - (g_1 - g_2)\|$ is convergent and $\lim\|s_2 - s_3 - (g_1 - g_2)\| = 0$.

(29) If s_1 is convergent and $\lim s_1 = g$, then $\|z \cdot s_1\|$ is convergent and $\lim\|z \cdot s_1\| = \|z \cdot g\|$.

(30) If s_1 is convergent and $\lim s_1 = g$, then $\|z \cdot s_1 - z \cdot g\|$ is convergent and $\lim\|z \cdot s_1 - z \cdot g\| = 0$.

(31) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1\|$ is convergent and $\lim\|-s_1\| = \|-g\|$.

(32) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1 - -g\|$ is convergent and $\lim\|-s_1 - -g\| = 0$.

(33) If s_1 is convergent and $\lim s_1 = g$, then $\|(s_1+x)-(g+x)\|$ is convergent and $\lim\|(s_1 + x) - (g + x)\| = 0$.

(34) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x\|$ is convergent and $\lim\|s_1 - x\| = \|g - x\|$.

(35) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x - (g - x)\|$ is convergent and $\lim\|s_1 - x - (g - x)\| = 0$.

(36) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 + s_3, g_1 + g_2)$ is convergent and $\lim \rho(s_2 + s_3, g_1 + g_2) = 0$.

(37) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 - s_3, g_1 - g_2)$ is convergent and $\lim \rho(s_2 - s_3, g_1 - g_2) = 0$.

(38) If s_1 is convergent and $\lim s_1 = g$, then $\rho(z \cdot s_1, z \cdot g)$ is convergent and $\lim \rho(z \cdot s_1, z \cdot g) = 0$.

(39) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1+x, g+x)$ is convergent and $\lim \rho(s_1 + x, g + x) = 0$.

Let us consider X, x, r. The functor $\mathrm{Ball}(x, r)$ yields a subset of X and is defined by:

(Def. 5) $\mathrm{Ball}(x, r) = \{y; y$ ranges over points of X: $\|x - y\| < r\}$.

The functor $\overline{\mathrm{Ball}}(x, r)$ yielding a subset of X is defined by:

(Def. 6) $\overline{\mathrm{Ball}}(x, r) = \{y; y$ ranges over points of X: $\|x - y\| \leqslant r\}$.

The functor $\mathrm{Sphere}(x, r)$ yielding a subset of X is defined as follows:

(Def. 7) $\mathrm{Sphere}(x, r) = \{y; y$ ranges over points of X: $\|x - y\| = r\}$.

Next we state a number of propositions:

(40) $w \in \mathrm{Ball}(x,r)$ iff $\|x-w\| < r$.

(41) $w \in \mathrm{Ball}(x,r)$ iff $\rho(x,w) < r$.

(42) If $r > 0$, then $x \in \mathrm{Ball}(x,r)$.

(43) If $y \in \mathrm{Ball}(x,r)$ and $w \in \mathrm{Ball}(x,r)$, then $\rho(y,w) < 2 \cdot r$.

(44) If $y \in \mathrm{Ball}(x,r)$, then $y - w \in \mathrm{Ball}(x-w,r)$.

(45) If $y \in \mathrm{Ball}(x,r)$, then $y - x \in \mathrm{Ball}(0_X,r)$.

(46) If $y \in \mathrm{Ball}(x,r)$ and $r \leqslant q$, then $y \in \mathrm{Ball}(x,q)$.

(47) $w \in \overline{\mathrm{Ball}}(x,r)$ iff $\|x-w\| \leqslant r$.

(48) $w \in \overline{\mathrm{Ball}}(x,r)$ iff $\rho(x,w) \leqslant r$.

(49) If $r \geqslant 0$, then $x \in \overline{\mathrm{Ball}}(x,r)$.

(50) If $y \in \mathrm{Ball}(x,r)$, then $y \in \overline{\mathrm{Ball}}(x,r)$.

(51) $w \in \mathrm{Sphere}(x,r)$ iff $\|x-w\| = r$.

(52) $w \in \mathrm{Sphere}(x,r)$ iff $\rho(x,w) = r$.

(53) If $y \in \mathrm{Sphere}(x,r)$, then $y \in \overline{\mathrm{Ball}}(x,r)$.

(54) $\mathrm{Ball}(x,r) \subseteq \overline{\mathrm{Ball}}(x,r)$.

(55) $\mathrm{Sphere}(x,r) \subseteq \overline{\mathrm{Ball}}(x,r)$.

(56) $\mathrm{Ball}(x,r) \cup \mathrm{Sphere}(x,r) = \overline{\mathrm{Ball}}(x,r)$.

2. Cauchy Sequence and Hilbert Space with Complex Coefficient

Let us consider X and let us consider s_1. We say that s_1 is Cauchy if and only if:

(Def. 8) For every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\rho(s_1(n), s_1(m)) < r$.

The following propositions are true:

(57) If s_1 is constant, then s_1 is Cauchy.

(58) s_1 is Cauchy if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|s_1(n) - s_1(m)\| < r$.

(59) If s_2 is Cauchy and s_3 is Cauchy, then $s_2 + s_3$ is Cauchy.

(60) If s_2 is Cauchy and s_3 is Cauchy, then $s_2 - s_3$ is Cauchy.

(61) If s_1 is Cauchy, then $z \cdot s_1$ is Cauchy.

(62) If s_1 is Cauchy, then $-s_1$ is Cauchy.

(63) If s_1 is Cauchy, then $s_1 + x$ is Cauchy.

(64) If s_1 is Cauchy, then $s_1 - x$ is Cauchy.

(65) If s_1 is convergent, then s_1 is Cauchy.

Let us consider X and let us consider s_2, s_3. We say that s_2 is compared to s_3 if and only if:

(Def. 9) For every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\rho(s_2(n), s_3(n)) < r$.

One can prove the following two propositions:

(66) s_1 is compared to s_1.

(67) If s_2 is compared to s_3, then s_3 is compared to s_2.

Let us consider X and let us consider s_2, s_3. Let us notice that the predicate s_2 is compared to s_3 is reflexive and symmetric.

The following propositions are true:

(68) If s_2 is compared to s_3 and s_3 is compared to s_4, then s_2 is compared to s_4.

(69) s_2 is compared to s_3 iff for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\|s_2(n) - s_3(n)\| < r$.

(70) If there exists k such that for every n such that $n \geqslant k$ holds $s_2(n) = s_3(n)$, then s_2 is compared to s_3.

(71) If s_2 is Cauchy and compared to s_3, then s_3 is Cauchy.

(72) If s_2 is convergent and compared to s_3, then s_3 is convergent.

(73) If s_2 is convergent and $\lim s_2 = g$ and s_2 is compared to s_3, then s_3 is convergent and $\lim s_3 = g$.

Let us consider X and let us consider s_1. We say that s_1 is bounded if and only if:

(Def. 10) There exists M such that $M > 0$ and for every n holds $\|s_1(n)\| \leqslant M$.

We now state several propositions:

(74) If s_2 is bounded and s_3 is bounded, then $s_2 + s_3$ is bounded.

(75) If s_1 is bounded, then $-s_1$ is bounded.

(76) If s_2 is bounded and s_3 is bounded, then $s_2 - s_3$ is bounded.

(77) If s_1 is bounded, then $z \cdot s_1$ is bounded.

(78) If s_1 is constant, then s_1 is bounded.

(79) For every m there exists M such that $M > 0$ and for every n such that $n \leqslant m$ holds $\|s_1(n)\| < M$.

(80) If s_1 is convergent, then s_1 is bounded.

(81) If s_2 is bounded and compared to s_3, then s_3 is bounded.

Let us consider X, N_1, s_1. Then $s_1 \cdot N_1$ is a sequence of X.

We now state several propositions:

(82) Let X be a complex unitary space, s be a sequence of X, N be an increasing sequence of naturals, and n be a natural number. Then $(s \cdot N)(n) = s(N(n))$.

(83) s_1 is a subsequence of s_1.

(84) If s_2 is a subsequence of s_3 and s_3 is a subsequence of s_4, then s_2 is a subsequence of s_4.

(85) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(86) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

(87) If s_1 is bounded and s_2 is a subsequence of s_1, then s_2 is bounded.

(88) If s_1 is convergent and s_2 is a subsequence of s_1, then s_2 is convergent.

(89) If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(90) If s_1 is Cauchy and s_2 is a subsequence of s_1, then s_2 is Cauchy.

Let us consider X, let us consider s_1, and let us consider k. The functor $s_1 \uparrow k$ yields a sequence of X and is defined as follows:

(Def. 11) For every n holds $(s_1 \uparrow k)(n) = s_1(n + k)$.

One can prove the following propositions:

(91) $s_1 \uparrow 0 = s_1$.

(92) $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(93) $s_1 \uparrow k \uparrow m = s_1 \uparrow (k + m)$.

(94) $(s_2 + s_3) \uparrow k = s_2 \uparrow k + s_3 \uparrow k$.

(95) $(-s_1) \uparrow k = -s_1 \uparrow k$.

(96) $(s_2 - s_3) \uparrow k = s_2 \uparrow k - s_3 \uparrow k$.

(97) $(z \cdot s_1) \uparrow k = z \cdot (s_1 \uparrow k)$.

(98) $(s_1 \cdot N_1) \uparrow k = s_1 \cdot (N_1 \uparrow k)$.

(99) $s_1 \uparrow k$ is a subsequence of s_1.

(100) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(101) If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is convergent.

(102) If s_1 is Cauchy and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is Cauchy.

(103) If s_1 is Cauchy, then $s_1 \uparrow k$ is Cauchy.

(104) If s_2 is compared to s_3, then $s_2 \uparrow k$ is compared to $s_3 \uparrow k$.

(105) If s_1 is bounded, then $s_1 \uparrow k$ is bounded.

(106) If s_1 is constant, then $s_1 \uparrow k$ is constant.

Let us consider X. We say that X is complete if and only if:

(Def. 12) For every s_1 such that s_1 is Cauchy holds s_1 is convergent.

The following proposition is true

(107) If X is complete and s_1 is Cauchy, then s_1 is bounded.

Let us consider X. We say that X is Hilbert if and only if:

(Def. 13) X is a complex unitary space and complete.

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Recursive Definitions. Part II[1]

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The papers [7], [4], [9], [8], [5], [6], [1], [10], [2], [11], and [3] provide the terminology and notation for this paper.

In this paper a, b, c, d, e, z, A, B, C, D, E are sets.

Let x be a set. Let us assume that there exist sets x_1, x_2, x_3 such that $x = \langle x_1, x_2, x_3 \rangle$. The functor $x_{\mathbf{1},3}$ is defined as follows:

(Def. 1) For all sets y_1, y_2, y_3 such that $x = \langle y_1, y_2, y_3 \rangle$ holds $x_{\mathbf{1},3} = y_1$.

The functor $x_{\mathbf{2},3}$ is defined by:

(Def. 2) For all sets y_1, y_2, y_3 such that $x = \langle y_1, y_2, y_3 \rangle$ holds $x_{\mathbf{2},3} = y_2$.

The functor $x_{\mathbf{3},3}$ is defined by:

(Def. 3) For all sets y_1, y_2, y_3 such that $x = \langle y_1, y_2, y_3 \rangle$ holds $x_{\mathbf{3},3} = y_3$.

The following propositions are true:

(1) If there exist a, b, c such that $z = \langle a, b, c \rangle$, then $z = \langle z_{\mathbf{1},3}, z_{\mathbf{2},3}, z_{\mathbf{3},3} \rangle$.

(2) If $z \in [\!: A, B, C :\!]$, then $z_{\mathbf{1},3} \in A$ and $z_{\mathbf{2},3} \in B$ and $z_{\mathbf{3},3} \in C$.

(3) If $z \in [\!: A, B, C :\!]$, then $z = \langle z_{\mathbf{1},3}, z_{\mathbf{2},3}, z_{\mathbf{3},3} \rangle$.

Let x be a set. Let us assume that there exist sets x_1, x_2, x_3, x_4 such that $x = \langle x_1, x_2, x_3, x_4 \rangle$. The functor $x_{\mathbf{1},4}$ is defined by:

(Def. 4) For all sets y_1, y_2, y_3, y_4 such that $x = \langle y_1, y_2, y_3, y_4 \rangle$ holds $x_{\mathbf{1},4} = y_1$.

The functor $x_{\mathbf{2},4}$ is defined by:

(Def. 5) For all sets y_1, y_2, y_3, y_4 such that $x = \langle y_1, y_2, y_3, y_4 \rangle$ holds $x_{\mathbf{2},4} = y_2$.

The functor $x_{\mathbf{3},4}$ is defined as follows:

(Def. 6) For all sets y_1, y_2, y_3, y_4 such that $x = \langle y_1, y_2, y_3, y_4 \rangle$ holds $x_{\mathbf{3},4} = y_3$.

The functor $x_{\mathbf{4},4}$ is defined as follows:

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(Def. 7) For all sets y_1, y_2, y_3, y_4 such that $x = \langle y_1, y_2, y_3, y_4 \rangle$ holds $x_{\mathbf{4},4} = y_4$.

Next we state three propositions:

(4) If there exist a, b, c, d such that $z = \langle a, b, c, d \rangle$, then $z = \langle z_{\mathbf{1},4}, z_{\mathbf{2},4}, z_{\mathbf{3},4}, z_{\mathbf{4},4} \rangle$.

(5) If $z \in [\!: A, B, C, D :\!]$, then $z_{\mathbf{1},4} \in A$ and $z_{\mathbf{2},4} \in B$ and $z_{\mathbf{3},4} \in C$ and $z_{\mathbf{4},4} \in D$.

(6) If $z \in [\!: A, B, C, D :\!]$, then $z = \langle z_{\mathbf{1},4}, z_{\mathbf{2},4}, z_{\mathbf{3},4}, z_{\mathbf{4},4} \rangle$.

Let x be a set. Let us assume that there exist sets x_1, x_2, x_3, x_4, x_5 such that $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$. The functor $x_{\mathbf{1},5}$ is defined by:

(Def. 8) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{1},5} = y_1$.

The functor $x_{\mathbf{2},5}$ is defined by:

(Def. 9) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{2},5} = y_2$.

The functor $x_{\mathbf{3},5}$ is defined as follows:

(Def. 10) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{3},5} = y_3$.

The functor $x_{\mathbf{4},5}$ is defined as follows:

(Def. 11) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{4},5} = y_4$.

The functor $x_{\mathbf{5},5}$ is defined by:

(Def. 12) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{5},5} = y_5$.

The following propositions are true:

(7) If there exist a, b, c, d, e such that $z = \langle a, b, c, d, e \rangle$, then $z = \langle z_{\mathbf{1},5}, z_{\mathbf{2},5}, z_{\mathbf{3},5}, z_{\mathbf{4},5}, z_{\mathbf{5},5} \rangle$.

(8) If $z \in [\!: A, B, C, D, E :\!]$, then $z_{\mathbf{1},5} \in A$ and $z_{\mathbf{2},5} \in B$ and $z_{\mathbf{3},5} \in C$ and $z_{\mathbf{4},5} \in D$ and $z_{\mathbf{5},5} \in E$.

(9) If $z \in [\!: A, B, C, D, E :\!]$, then $z = \langle z_{\mathbf{1},5}, z_{\mathbf{2},5}, z_{\mathbf{3},5}, z_{\mathbf{4},5}, z_{\mathbf{5},5} \rangle$.

In this article we present several logical schemes. The scheme *ExFunc3Cond* deals with a set $\mathcal{A}$, three unary functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ yielding sets, and three unary predicates $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and states that:

> There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and for every set c such that $c \in \mathcal{A}$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ and if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ and if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$

provided the parameters meet the following conditions:

- For every set c such that $c \in \mathcal{A}$ holds if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$ and if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$ and if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$, and
- For every set c such that $c \in \mathcal{A}$ holds $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$.

The scheme *ExFunc4Cond* deals with a set $\mathcal{A}$, four unary functors $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ yielding sets, and four unary predicates $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, and states that:

There exists a function f such that

(i) $\operatorname{dom} f = \mathcal{A}$, and

(ii) for every set c such that $c \in \mathcal{A}$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ and if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ and if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$ and if $\mathcal{S}[c]$, then $f(c) = \mathcal{I}(c)$

provided the following conditions are satisfied:

- Let c be a set such that $c \in \mathcal{A}$. Then
  (i) if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$,
  (ii) if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$,
  (iii) if $\mathcal{P}[c]$, then not $\mathcal{S}[c]$,
  (iv) if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$,
  (v) if $\mathcal{Q}[c]$, then not $\mathcal{S}[c]$, and
  (vi) if $\mathcal{R}[c]$, then not $\mathcal{S}[c]$,
  and
- For every set c such that $c \in \mathcal{A}$ holds $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$ or $\mathcal{S}[c]$.

The scheme *DoubleChoiceRec* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{B}$, and a 5-ary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ and there exists a function g from $\mathbb{N}$ into $\mathcal{B}$ such that $f(0) = \mathcal{C}$ and $g(0) = \mathcal{D}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), g(n), f(n+1), g(n+1)]$

provided the parameters satisfy the following condition:

- Let n be an element of $\mathbb{N}$, x be an element of $\mathcal{A}$, and y be an element of $\mathcal{B}$. Then there exists an element x_1 of $\mathcal{A}$ and there exists an element y_1 of $\mathcal{B}$ such that $\mathcal{P}[n, x, y, x_1, y_1]$.

The scheme *LambdaRec2Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a ternary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and $f(1) = \mathcal{B}$ and for every natural number n holds $f(n+2) = \mathcal{F}(n, f(n), f(n+1))$

for all values of the parameters.

The scheme *LambdaRec2ExD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and $f(1) = \mathcal{C}$ and for every natural number n holds $f(n+2) = \mathcal{F}(n, f(n), f(n+1))$

for all values of the parameters.

The scheme *LambdaRec2Un* deals with sets $\mathcal{A}$, $\mathcal{B}$, functions $\mathcal{C}$, $\mathcal{D}$, and a ternary functor $\mathcal{C}$ yielding a set, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters meet the following requirements:

- $\operatorname{dom} \mathcal{C} = \mathbb{N}$,
- $\mathcal{C}(0) = \mathcal{A}$ and $\mathcal{C}(1) = \mathcal{B}$,
- For every natural number n holds $\mathcal{C}(n+2) = \mathcal{C}(n, \mathcal{C}(n), \mathcal{C}(n+1))$,
- $\operatorname{dom} \mathcal{D} = \mathbb{N}$,
- $\mathcal{D}(0) = \mathcal{A}$ and $\mathcal{D}(1) = \mathcal{B}$, and
- For every natural number n holds $\mathcal{D}(n+2) = \mathcal{C}(n, \mathcal{D}(n), \mathcal{D}(n+1))$.

The scheme *LambdaRec2UnD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, functions $\mathcal{D}$, $\mathcal{E}$ from $\mathbb{N}$ into $\mathcal{A}$, and a ternary functor $\mathcal{D}$ yielding an element of $\mathcal{A}$, and states that:

$\mathcal{D} = \mathcal{E}$

provided the following requirements are met:

- $\mathcal{D}(0) = \mathcal{B}$ and $\mathcal{D}(1) = \mathcal{C}$,
- For every natural number n holds $\mathcal{D}(n+2) = \mathcal{D}(n, \mathcal{D}(n), \mathcal{D}(n+1))$,
- $\mathcal{E}(0) = \mathcal{B}$ and $\mathcal{E}(1) = \mathcal{C}$, and
- For every natural number n holds $\mathcal{E}(n+2) = \mathcal{D}(n, \mathcal{E}(n), \mathcal{E}(n+1))$.

The scheme *LambdaRec3Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a 4-ary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and $f(1) = \mathcal{B}$ and $f(2) = \mathcal{C}$ and for every natural number n holds $f(n+3) = \mathcal{F}(n, f(n), f(n+1), f(n+2))$

for all values of the parameters.

The scheme *LambdaRec3ExD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ of $\mathcal{A}$, and a 4-ary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and $f(1) = \mathcal{C}$ and $f(2) = \mathcal{D}$ and for every natural number n holds $f(n+3) = \mathcal{F}(n, f(n), f(n+1), f(n+2))$

for all values of the parameters.

The scheme *LambdaRec3Un* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, functions $\mathcal{D}$, $\mathcal{E}$, and a 4-ary functor $\mathcal{D}$ yielding a set, and states that:

$\mathcal{D} = \mathcal{E}$

provided the parameters meet the following requirements:

- $\operatorname{dom} \mathcal{D} = \mathbb{N}$,
- $\mathcal{D}(0) = \mathcal{A}$ and $\mathcal{D}(1) = \mathcal{B}$ and $\mathcal{D}(2) = \mathcal{C}$,
- For every natural number n holds $\mathcal{D}(n+3) = \mathcal{D}(n, \mathcal{D}(n), \mathcal{D}(n+1), \mathcal{D}(n+2))$,
- $\operatorname{dom} \mathcal{E} = \mathbb{N}$,
- $\mathcal{E}(0) = \mathcal{A}$ and $\mathcal{E}(1) = \mathcal{B}$ and $\mathcal{E}(2) = \mathcal{C}$, and
- For every natural number n holds $\mathcal{E}(n+3) = \mathcal{D}(n, \mathcal{E}(n), \mathcal{E}(n+1), \mathcal{E}(n+2))$.

The scheme *LambdaRec3UnD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ of $\mathcal{A}$, functions $\mathcal{E}$, $\mathcal{F}$ from $\mathbb{N}$ into $\mathcal{A}$, and a 4-ary functor $\mathcal{E}$ yielding an element of $\mathcal{A}$, and states that:

$\mathcal{E} = \mathcal{F}$

provided the parameters meet the following requirements:

- $\mathcal{E}(0) = \mathcal{B}$ and $\mathcal{E}(1) = \mathcal{C}$ and $\mathcal{E}(2) = \mathcal{D}$,
- For every natural number n holds $\mathcal{E}(n+3) = \mathcal{E}(n, \mathcal{E}(n), \mathcal{E}(n+1), \mathcal{E}(n+2))$,
- $\mathcal{F}(0) = \mathcal{B}$ and $\mathcal{F}(1) = \mathcal{C}$ and $\mathcal{F}(2) = \mathcal{D}$, and
- For every natural number n holds $\mathcal{F}(n+3) = \mathcal{E}(n, \mathcal{F}(n), \mathcal{F}(n+1), \mathcal{F}(n+2))$.

The scheme *LambdaRec4Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and a 5-ary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and $f(1) = \mathcal{B}$ and $f(2) = \mathcal{C}$ and $f(3) = \mathcal{D}$ and for every natural number n holds $f(n+4) = \mathcal{F}(n, f(n), f(n+1), f(n+2), f(n+3))$

for all values of the parameters.

The scheme *LambdaRec4ExD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ of $\mathcal{A}$, and a 5-ary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

> There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and $f(1) = \mathcal{C}$ and $f(2) = \mathcal{D}$ and $f(3) = \mathcal{E}$ and for every natural number n holds $f(n+4) = \mathcal{F}(n, f(n), f(n+1), f(n+2), f(n+3))$

for all values of the parameters.

The scheme *LambdaRec4Un* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, functions $\mathcal{E}$, $\mathcal{F}$, and a 5-ary functor $\mathcal{E}$ yielding a set, and states that:

$\mathcal{E} = \mathcal{F}$

provided the parameters satisfy the following conditions:

- $\operatorname{dom} \mathcal{E} = \mathbb{N}$,
- $\mathcal{E}(0) = \mathcal{A}$ and $\mathcal{E}(1) = \mathcal{B}$ and $\mathcal{E}(2) = \mathcal{C}$ and $\mathcal{E}(3) = \mathcal{D}$,
- For every natural number n holds $\mathcal{E}(n+4) = \mathcal{E}(n, \mathcal{E}(n), \mathcal{E}(n+1), \mathcal{E}(n+2), \mathcal{E}(n+3))$,
- $\operatorname{dom} \mathcal{F} = \mathbb{N}$,
- $\mathcal{F}(0) = \mathcal{A}$ and $\mathcal{F}(1) = \mathcal{B}$ and $\mathcal{F}(2) = \mathcal{C}$ and $\mathcal{F}(3) = \mathcal{D}$, and
- For every natural number n holds $\mathcal{F}(n+4) = \mathcal{E}(n, \mathcal{F}(n), \mathcal{F}(n+1), \mathcal{F}(n+2), \mathcal{F}(n+3))$.

The scheme *LambdaRec4UnD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ of $\mathcal{A}$, functions $\mathcal{F}$, $\mathcal{G}$ from $\mathbb{N}$ into $\mathcal{A}$, and a 5-ary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

$\mathcal{F} = \mathcal{G}$

provided the parameters meet the following requirements:

- $\mathcal{F}(0) = \mathcal{B}$ and $\mathcal{F}(1) = \mathcal{C}$ and $\mathcal{F}(2) = \mathcal{D}$ and $\mathcal{F}(3) = \mathcal{E}$,
- For every natural number n holds $\mathcal{F}(n+4) = \mathcal{F}(n, \mathcal{F}(n), \mathcal{F}(n+1), \mathcal{F}(n+2), \mathcal{F}(n+3))$,
- $\mathcal{G}(0) = \mathcal{B}$ and $\mathcal{G}(1) = \mathcal{C}$ and $\mathcal{G}(2) = \mathcal{D}$ and $\mathcal{G}(3) = \mathcal{E}$, and

- For every natural number n holds $\mathcal{G}(n+4) = \mathcal{F}(n, \mathcal{G}(n), \mathcal{G}(n+1), \mathcal{G}(n+2), \mathcal{G}(n+3))$.

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The Exponential Function on Banach Algebra

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Summary. In this article, the basic properties of the exponential function on Banach algebra are described.

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The notation and terminology used here are introduced in the following papers: [17], [19], [20], [3], [4], [2], [16], [5], [1], [18], [9], [11], [12], [8], [6], [7], [13], [10], [21], [14], and [15].

For simplicity, we use the following convention: X denotes a Banach algebra, p denotes a real number, w, z, z_1, z_2 denote elements of X, k, l, m, n denote natural numbers, s_1, s_2, s_3, s, s' denote sequences of X, and r_1 denotes a sequence of real numbers.

Let X be a non empty normed algebra structure and let x, y be elements of X. We say that x, y are commutative if and only if:

(Def. 1) $x \cdot y = y \cdot x$.

Let us note that the predicate x, y are commutative is symmetric.

Next we state a number of propositions:

(1) If s_2 is convergent and s_3 is convergent and $\lim(s_2 - s_3) = 0_X$, then $\lim s_2 = \lim s_3$.

(2) For every z such that for every natural number n holds $s(n) = z$ holds $\lim s = z$.

(3) If s is convergent and s' is convergent, then $s \cdot s'$ is convergent.

(4) If s is convergent, then $z \cdot s$ is convergent.

(5) If s is convergent, then $s \cdot z$ is convergent.

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(6) If s is convergent, then $\lim(z \cdot s) = z \cdot \lim s$.

(7) If s is convergent, then $\lim(s \cdot z) = \lim s \cdot z$.

(8) If s is convergent and s' is convergent, then $\lim(s \cdot s') = \lim s \cdot \lim s'$.

(9) $(\sum_{\alpha=0}^{\kappa}(z \cdot s_1)(\alpha))_{\kappa \in \mathbb{N}} = z \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}$ and $(\sum_{\alpha=0}^{\kappa}(s_1 \cdot z)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}} \cdot z$.

(10) $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(k)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa \in \mathbb{N}}(k)$.

(11) If for every n such that $n \leqslant m$ holds $s_2(n) = s_3(n)$, then $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa \in \mathbb{N}}(m) = (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(12) If for every n holds $\|s_1(n)\| \leqslant r_1(n)$ and r_1 is convergent and $\lim r_1 = 0$, then s_1 is convergent and $\lim s_1 = 0_X$.

Let us consider X and let z be an element of X. The functor $z\,\mathrm{ExpSeq}$ yielding a sequence of X is defined as follows:

(Def. 2) For every n holds $z\,\mathrm{ExpSeq}(n) = \frac{1}{n!} \cdot z_{\mathbb{N}}^{n}$.

The scheme *ExNormSpace CASE* deals with a non empty Banach algebra $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

For every k there exists a sequence s_1 of $\mathcal{A}$ such that for every n holds if $n \leqslant k$, then $s_1(n) = \mathcal{F}(k, n)$ and if $n > k$, then $s_1(n) = 0_{\mathcal{A}}$

for all values of the parameters.

Next we state the proposition

(13) For every k such that $0 < k$ holds $(k -' 1)! \cdot k = k!$ and for all m, k such that $k \leqslant m$ holds $(m -' k)! \cdot ((m + 1) - k) = ((m + 1) -' k)!$.

Let n be a natural number. The functor $\mathrm{Coef}\,n$ yields a sequence of real numbers and is defined by:

(Def. 3) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Coef}\,n)(k) = \frac{n!}{k! \cdot (n -' k)!}$ and if $k > n$, then $(\mathrm{Coef}\,n)(k) = 0$.

Let n be a natural number. The functor $\mathrm{Coef\_e}\,n$ yielding a sequence of real numbers is defined by:

(Def. 4) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Coef\_e}\,n)(k) = \frac{1}{k! \cdot (n -' k)!}$ and if $k > n$, then $(\mathrm{Coef\_e}\,n)(k) = 0$.

Let us consider X, s_1. The functor $\mathrm{Shift}\,s_1$ yielding a sequence of X is defined as follows:

(Def. 5) $(\mathrm{Shift}\,s_1)(0) = 0_X$ and for every natural number k holds $(\mathrm{Shift}\,s_1)(k + 1) = s_1(k)$.

Let us consider n, let us consider X, and let z, w be elements of X. The functor $\mathrm{Expan}(n, z, w)$ yields a sequence of X and is defined by:

(Def. 6) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan}(n, z, w))(k) = (\mathrm{Coef}\,n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{n -' k}$ and if $n < k$, then $(\mathrm{Expan}(n, z, w))(k) = 0_X$.

Let us consider n, let us consider X, and let z, w be elements of X. The functor $\mathrm{Expan\_e}(n, z, w)$ yields a sequence of X and is defined as follows:

(Def. 7) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan\_e}(n, z, w))(k) = (\mathrm{Coef\_e}\, n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{n -' k}$ and if $n < k$, then $(\mathrm{Expan\_e}(n, z, w))(k) = 0_X$.

Let us consider n, let us consider X, and let z, w be elements of X. The functor $\mathrm{Alfa}(n, z, w)$ yields a sequence of X and is defined as follows:

(Def. 8) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Alfa}(n, z, w))(k) = z\,\mathrm{ExpSeq}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n -' k)$ and if $n < k$, then $(\mathrm{Alfa}(n, z, w))(k) = 0_X$.

Let us consider X, let z, w be elements of X, and let n be a natural number. The functor $\mathrm{Conj}(n, z, w)$ yields a sequence of X and is defined by:

(Def. 9) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Conj}(n, z, w))(k) = z\,\mathrm{ExpSeq}(k) \cdot ((\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n -' k))$ and if $n < k$, then $(\mathrm{Conj}(n, z, w))(k) = 0_X$.

One can prove the following propositions:

(14) $z\,\mathrm{ExpSeq}(n + 1) = \frac{1}{n+1} \cdot z \cdot z\,\mathrm{ExpSeq}(n)$ and $z\,\mathrm{ExpSeq}(0) = \mathbf{1}_X$ and $\|z\,\mathrm{ExpSeq}(n)\| \leqslant \|z\|\,\mathrm{ExpSeq}(n)$.

(15) If $0 < k$, then $(\mathrm{Shift}\, s_1)(k) = s_1(k -' 1)$.

(16) $(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa} (\mathrm{Shift}\, s_1)(\alpha))_{\kappa \in \mathbb{N}}(k) + s_1(k)$.

(17) For all z, w such that z, w are commutative holds $(z + w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa} (\mathrm{Expan}(n, z, w))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(18) $\mathrm{Expan\_e}(n, z, w) = \frac{1}{n!} \cdot \mathrm{Expan}(n, z, w)$.

(19) For all z, w such that z, w are commutative holds $\frac{1}{n!} \cdot (z + w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa} (\mathrm{Expan\_e}(n, z, w))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(20) $0_X\,\mathrm{ExpSeq}$ is norm-summable and $\sum(0_X\,\mathrm{ExpSeq}) = \mathbf{1}_X$.

Let us consider X and let z be an element of X. Observe that $z\,\mathrm{ExpSeq}$ is norm-summable.

Next we state a number of propositions:

(21) $z\,\mathrm{ExpSeq}(0) = \mathbf{1}_X$ and $(\mathrm{Expan}(0, z, w))(0) = \mathbf{1}_X$.

(22) If $l \leqslant k$, then $(\mathrm{Alfa}(k + 1, z, w))(l) = (\mathrm{Alfa}(k, z, w))(l) + (\mathrm{Expan\_e}(k + 1, z, w))(l)$.

(23) $(\sum_{\alpha=0}^{\kappa} (\mathrm{Alfa}(k+1, z, w))(\alpha))_{\kappa \in \mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa} (\mathrm{Alfa}(k, z, w))(\alpha))_{\kappa \in \mathbb{N}}(k) + (\sum_{\alpha=0}^{\kappa} (\mathrm{Expan\_e}(k + 1, z, w))(\alpha))_{\kappa \in \mathbb{N}}(k)$.

(24) $z\,\mathrm{ExpSeq}(k) = (\mathrm{Expan\_e}(k, z, w))(k)$.

(25) For all z, w such that z, w are commutative holds $(\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa} (\mathrm{Alfa}(n, z, w))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(26) For all z, w such that z, w are commutative holds $(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k) - (\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa \in \mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa} (\mathrm{Conj}(k, z, w))(\alpha))_{\kappa \in \mathbb{N}}(k)$.

(27) $0 \leqslant \|z\|\,\mathrm{ExpSeq}(n)$.

(28) $\|(\sum_{\alpha=0}^{\kappa} z \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)$ and $(\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) \leqslant \sum(\|z\| \operatorname{ExpSeq})$ and $\|(\sum_{\alpha=0}^{\kappa} z \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)\| \leqslant \sum(\|z\| \operatorname{ExpSeq})$.

(29) $1 \leqslant \sum(\|z\| \operatorname{ExpSeq})$.

(30) $|(\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$ and if $n \leqslant m$, then $|(\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(31) $|(\sum_{\alpha=0}^{\kappa}\|\operatorname{Conj}(k,z,w)\|(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|\operatorname{Conj}(k,z,w)\|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(32) For every real number p such that $p > 0$ there exists n such that for every k such that $n \leqslant k$ holds $|(\sum_{\alpha=0}^{\kappa}\|\operatorname{Conj}(k,z,w)\|(\alpha))_{\kappa\in\mathbb{N}}(k)| < p$.

(33) For every s_1 such that for every k holds $s_1(k) = (\sum_{\alpha=0}^{\kappa}(\operatorname{Conj}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$ holds s_1 is convergent and $\lim s_1 = 0_X$.

Let X be a Banach algebra. The functor $\exp X$ yielding a function from the carrier of X into the carrier of X is defined by:

(Def. 10) For every element z of the carrier of X holds $(\exp X)(z) = \sum(z \operatorname{ExpSeq})$.

Let us consider X, z. The functor $\exp z$ yields an element of X and is defined by:

(Def. 11) $\exp z = (\exp X)(z)$.

One can prove the following propositions:

(34) For every z holds $\exp z = \sum(z \operatorname{ExpSeq})$.

(35) Let given z_1, z_2. Suppose z_1, z_2 are commutative. Then $\exp(z_1 + z_2) = \exp z_1 \cdot \exp z_2$ and $\exp(z_2 + z_1) = \exp z_2 \cdot \exp z_1$ and $\exp(z_1 + z_2) = \exp(z_2 + z_1)$ and $\exp z_1$, $\exp z_2$ are commutative.

(36) For all z_1, z_2 such that z_1, z_2 are commutative holds $z_1 \cdot \exp z_2 = \exp z_2 \cdot z_1$.

(37) $\exp(0_X) = \mathbf{1}_X$.

(38) $\exp z \cdot \exp(-z) = \mathbf{1}_X$ and $\exp(-z) \cdot \exp z = \mathbf{1}_X$.

(39) $\exp z$ is invertible and $(\exp z)^{-1} = \exp(-z)$ and $\exp(-z)$ is invertible and $(\exp(-z))^{-1} = \exp z$.

(40) For every z and for all real numbers s, t holds $s \cdot z$, $t \cdot z$ are commutative.

(41) Let given z and s, t be real numbers. Then $\exp(s \cdot z) \cdot \exp(t \cdot z) = \exp((s+t) \cdot z)$ and $\exp(t \cdot z) \cdot \exp(s \cdot z) = \exp((t+s) \cdot z)$ and $\exp((s+t) \cdot z) = \exp((t+s) \cdot z)$ and $\exp(s \cdot z)$, $\exp(t \cdot z)$ are commutative.

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Fundamental Theorem of Arithmetic[1]

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Summary. We formalize the notion of the prime-power factorization of a natural number and prove the Fundamental Theorem of Arithmetic. We prove also how prime-power factorization can be used to compute: products, quotients, powers, greatest common divisors and least common multiples.

MML Identifier: `NAT_3`.

The notation and terminology used in this paper are introduced in the following papers: [25], [27], [12], [7], [3], [4], [1], [24], [13], [2], [19], [18], [28], [8], [9], [6], [16], [15], [11], [26], [22], [23], [10], [14], [20], [5], [21], and [17].

1. Preliminaries

We follow the rules: a, b, n denote natural numbers, r denotes a real number, and f denotes a finite sequence of elements of $\mathbb{R}$.

Let X be an empty set. Observe that $\operatorname{card} X$ is empty.

One can check that every binary relation which is natural-yielding is also real-yielding.

Let us mention that there exists a finite sequence which is natural-yielding.

Let a be a non empty natural number and let b be a natural number. Observe that a^b is non empty.

One can verify that every prime number is non empty.

In the sequel p denotes a prime number.

One can verify that Prime is infinite.

The following propositions are true:

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(1) For all natural numbers a, b, c, d such that $a \mid c$ and $b \mid d$ holds $a \cdot b \mid c \cdot d$.

(2) If $1 < a$, then $b \leqslant a^b$.

(3) If $a \neq 0$, then $n \mid n^a$.

(4) For all natural numbers i, j, m, n such that $i < j$ and $m^j \mid n$ holds $m^{i+1} \mid n$.

(5) If $p \mid a^b$, then $p \mid a$.

(6) For every prime number a such that $a \mid p^b$ holds $a = p$.

(7) For every finite sequence f of elements of $\mathbb{N}$ such that $a \in \operatorname{rng} f$ holds $a \mid \prod f$.

(8) For every finite sequence f of elements of Prime such that $p \mid \prod f$ holds $p \in \operatorname{rng} f$.

Let f be a real-yielding finite sequence and let a be a natural number. The functor f^a yielding a finite sequence is defined as follows:

(Def. 1) $\operatorname{len}(f^a) = \operatorname{len} f$ and for every set i such that $i \in \operatorname{dom}(f^a)$ holds $f^a(i) = f(i)^a$.

Let f be a real-yielding finite sequence and let a be a natural number. One can verify that f^a is real-yielding.

Let f be a natural-yielding finite sequence and let a be a natural number. Note that f^a is natural-yielding.

Let f be a finite sequence of elements of $\mathbb{R}$ and let a be a natural number. Then f^a is a finite sequence of elements of $\mathbb{R}$.

Let f be a finite sequence of elements of $\mathbb{N}$ and let a be a natural number. Then f^a is a finite sequence of elements of $\mathbb{N}$.

Next we state several propositions:

(9) $f^0 = \operatorname{len} f \mapsto 1$.

(10) $f^1 = f$.

(11) $(\varepsilon_{\mathbb{R}})^a = \varepsilon_{\mathbb{R}}$.

(12) $\langle r \rangle^a = \langle r^a \rangle$.

(13) $(f \frown \langle r \rangle)^a = (f^a) \frown \langle r \rangle^a$.

(14) $\prod(f^{b+1}) = \prod(f^b) \cdot \prod f$.

(15) $\prod(f^a) = (\prod f)^a$.

2. More about Bags

Let X be a set. Note that there exists a many sorted set indexed by X which is natural-yielding and finite-support.

Let X be a set, let b be a real-yielding many sorted set indexed by X, and let a be a natural number. The functor $a \cdot b$ yielding a many sorted set indexed by X is defined as follows:

(Def. 2) For every set i holds $(a \cdot b)(i) = a \cdot b(i)$.

Let X be a set, let b be a real-yielding many sorted set indexed by X, and let a be a natural number. One can verify that $a \cdot b$ is real-yielding.

Let X be a set, let b be a natural-yielding many sorted set indexed by X, and let a be a natural number. Note that $a \cdot b$ is natural-yielding.

Let X be a set and let b be a real-yielding many sorted set indexed by X. Note that $\mathrm{support}(0 \cdot b)$ is empty.

Next we state the proposition

(16) For every set X and for every real-yielding many sorted set b indexed by X such that $a \neq 0$ holds $\mathrm{support}\, b = \mathrm{support}(a \cdot b)$.

Let X be a set, let b be a real-yielding finite-support many sorted set indexed by X, and let a be a natural number. One can check that $a \cdot b$ is finite-support.

Let X be a set and let b_1, b_2 be real-yielding many sorted sets indexed by X. The functor $\min(b_1, b_2)$ yields a many sorted set indexed by X and is defined by:

(Def. 3) For every set i holds if $b_1(i) \leqslant b_2(i)$, then $(\min(b_1, b_2))(i) = b_1(i)$ and if $b_1(i) > b_2(i)$, then $(\min(b_1, b_2))(i) = b_2(i)$.

Let X be a set and let b_1, b_2 be real-yielding many sorted sets indexed by X. Note that $\min(b_1, b_2)$ is real-yielding.

Let X be a set and let b_1, b_2 be natural-yielding many sorted sets indexed by X. Observe that $\min(b_1, b_2)$ is natural-yielding.

We now state the proposition

(17) For every set X and for all real-yielding finite-support many sorted sets b_1, b_2 indexed by X holds $\mathrm{support}\min(b_1, b_2) \subseteq \mathrm{support}\, b_1 \cup \mathrm{support}\, b_2$.

Let X be a set and let b_1, b_2 be real-yielding finite-support many sorted sets indexed by X. Observe that $\min(b_1, b_2)$ is finite-support.

Let X be a set and let b_1, b_2 be real-yielding many sorted sets indexed by X. The functor $\max(b_1, b_2)$ yielding a many sorted set indexed by X is defined as follows:

(Def. 4) For every set i holds if $b_1(i) \leqslant b_2(i)$, then $(\max(b_1, b_2))(i) = b_2(i)$ and if $b_1(i) > b_2(i)$, then $(\max(b_1, b_2))(i) = b_1(i)$.

Let X be a set and let b_1, b_2 be real-yielding many sorted sets indexed by X. Observe that $\max(b_1, b_2)$ is real-yielding.

Let X be a set and let b_1, b_2 be natural-yielding many sorted sets indexed by X. One can check that $\max(b_1, b_2)$ is natural-yielding.

One can prove the following proposition

(18) For every set X and for all real-yielding finite-support many sorted sets b_1, b_2 indexed by X holds $\mathrm{support}\max(b_1, b_2) \subseteq \mathrm{support}\, b_1 \cup \mathrm{support}\, b_2$.

Let X be a set and let b_1, b_2 be real-yielding finite-support many sorted sets indexed by X. Observe that $\max(b_1, b_2)$ is finite-support.

Let A be a set and let b be a bag of A. The functor $\prod b$ yields a natural number and is defined by:

(Def. 5) There exists a finite sequence f of elements of $\mathbb{N}$ such that $\prod b = \prod f$ and $f = b \cdot \mathrm{CFS}(\mathrm{support}\, b)$.

Let A be a set and let b be a bag of A. Then $\prod b$ is a natural number.

One can prove the following proposition

(19) For every set X and for all bags a, b of X such that support a misses support b holds $\prod(a+b) = \prod a \cdot \prod b$.

Let X be a set, let b be a real-yielding many sorted set indexed by X, and let n be a non empty natural number. The functor b^n yielding a many sorted set indexed by X is defined by:

(Def. 6) $\mathrm{support}(b^n) = \mathrm{support}\, b$ and for every set i holds $b^n(i) = b(i)^n$.

Let X be a set, let b be a natural-yielding many sorted set indexed by X, and let n be a non empty natural number. One can verify that b^n is natural-yielding.

Let X be a set, let b be a real-yielding finite-support many sorted set indexed by X, and let n be a non empty natural number. Observe that b^n is finite-support.

The following proposition is true

(20) For every set A holds $\prod \mathrm{EmptyBag}\, A = 1$.

3. Multiplicity of a Divisor

Let n, d be natural numbers. Let us assume that $d \neq 1$ and $n \neq 0$. The functor d-count(n) yields a natural number and is defined by:

(Def. 7) $d^{d\text{-count}(n)} \mid n$ and $d^{d\text{-count}(n)+1} \nmid n$.

One can prove the following propositions:

(21) If $n \neq 1$, then n-count$(1) = 0$.

(22) If $1 < n$, then n-count$(n) = 1$.

(23) If $b \neq 0$ and $b < a$ and $a \neq 1$, then a-count$(b) = 0$.

(24) If $a \neq 1$ and $a \neq p$, then a-count$(p) = 0$.

(25) If $1 < b$, then b-count$(b^a) = a$.

(26) If $b \neq 1$ and $a \neq 0$ and $b \mid b^{b\text{-count}(a)}$, then $b \mid a$.

(27) If $b \neq 1$, then $a \neq 0$ and b-count$(a) = 0$ iff $b \nmid a$.

(28) For all non empty natural numbers a, b holds p-count$(a \cdot b) = p$-count$(a) + p$-count(b).

(29) For all non empty natural numbers a, b holds $p^{p\text{-count}(a\cdot b)} = p^{p\text{-count}(a)} \cdot p^{p\text{-count}(b)}$.

(30) For all non empty natural numbers a, b such that $b \mid a$ holds p-count$(b) \leqslant p$-count(a).

(31) For all non empty natural numbers a, b such that $b \mid a$ holds p-count$(a \div b) = p$-count$(a) -' p$-count(b).

(32) For every non empty natural number a holds p-count$(a^b) = b \cdot p$-count(a).

4. Exponents in Prime-Power Factorization

Let n be a natural number. The functor PrimeExponents(n) yields a many sorted set indexed by Prime and is defined as follows:

(Def. 8) For every prime number p holds $(\text{PrimeExponents}(n))(p) = p$-count$(n)$.

We introduce PFExp(n) as a synonym of PrimeExponents(n).

One can prove the following three propositions:

(33) For every set x such that $x \in \operatorname{dom} \text{PFExp}(n)$ holds x is a prime number.

(34) For every set x such that $x \in \operatorname{support} \text{PFExp}(n)$ holds x is a prime number.

(35) If $a > n$ and $n \neq 0$, then $(\text{PFExp}(n))(a) = 0$.

Let n be a natural number. Note that PFExp(n) is natural-yielding.

One can prove the following two propositions:

(36) If $a \in \operatorname{support} \text{PFExp}(b)$, then $a \mid b$.

(37) If b is non empty and a is a prime number and $a \mid b$, then $a \in \operatorname{support} \text{PFExp}(b)$.

Let n be a non empty natural number. Observe that PFExp(n) is finite-support.

We now state two propositions:

(38) For every non empty natural number a such that $p \mid a$ holds $(\text{PFExp}(a))(p) \neq 0$.

(39) $\text{PFExp}(1) = \text{EmptyBag}\,\text{Prime}$.

One can verify that support PFExp(1) is empty.

One can prove the following four propositions:

(40) $(\text{PFExp}(p^a))(p) = a$.

(41) $(\text{PFExp}(p))(p) = 1$.

(42) If $a \neq 0$, then $\operatorname{support} \text{PFExp}(p^a) = \{p\}$.

(43) $\operatorname{support} \text{PFExp}(p) = \{p\}$.

Let p be a prime number and let a be a non empty natural number. Observe that support PFExp(p^a) is non empty and trivial.

Let p be a prime number. Observe that support PFExp(p) is non empty and trivial.

Next we state several propositions:

(44) For all non empty natural numbers a, b such that a and b are relative prime holds support PFExp(a) misses support PFExp(b).

(45) For all non empty natural numbers a, b holds support PFExp$(a) \subseteq$ support PFExp$(a \cdot b)$.

(46) For all non empty natural numbers a, b holds support PFExp$(a \cdot b) =$ support PFExp$(a) \cup$ support PFExp(b).

(47) For all non empty natural numbers a, b such that a and b are relative prime holds card support PFExp$(a \cdot b) =$ card support PFExp$(a) +$ card support PFExp(b).

(48) For all non empty natural numbers a, b holds support PFExp$(a) =$ support PFExp(a^b).

In the sequel n, m are non empty natural numbers.

Next we state several propositions:

(49) PFExp$(n \cdot m) =$ PFExp$(n) +$ PFExp(m).

(50) If $m \mid n$, then PFExp$(n \div m) =$ PFExp$(n) -'$ PFExp(m).

(51) PFExp$(n^a) = a \cdot$ PFExp(n).

(52) If support PFExp$(n) = \emptyset$, then $n = 1$.

(53) For all non empty natural numbers m, n holds PFExp$(\gcd(n, m)) = \min($PFExp$(n),$ PFExp$(m))$.

(54) For all non empty natural numbers m, n holds PFExp$(\mathrm{lcm}(n, m)) = \max($PFExp$(n),$ PFExp$(m))$.

5. Prime-Power Factorization

Let n be a non empty natural number. The functor PrimeFactorization(n) yielding a many sorted set indexed by Prime is defined as follows:

(Def. 9) support PrimeFactorization$(n) =$ support PFExp(n) and for every natural number p such that $p \in$ support PFExp(n) holds (PrimeFactorization$(n))(p) = p^{p\text{-count}(n)}$.

We introduce PPF(n) as a synonym of PrimeFactorization(n).

Let n be a non empty natural number. Observe that PPF(n) is natural-yielding and finite-support.

The following propositions are true:

(55) If p-count$(n) = 0$, then $(\mathrm{PPF}(n))(p) = 0$.

(56) If p-count$(n) \neq 0$, then $(\mathrm{PPF}(n))(p) = p^{p\text{-count}(n)}$.

(57) If support PPF$(n) = \emptyset$, then $n = 1$.

(58) For all non empty natural numbers a, b such that a and b are relative prime holds PPF$(a \cdot b) =$ PPF$(a) +$ PPF(b).

(59) $(\mathrm{PPF}(p^n))(p) = p^n$.

(60) $\mathrm{PPF}(n^m) = (\mathrm{PPF}(n))^m$.

(61) $\prod \mathrm{PPF}(n) = n$.

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Hilbert Space of Complex Sequences

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Summary. An extension of [9]. As the example of complex norm spaces, we introduce the arithmetic addition and multiplication in the set of absolute summable complex sequences and also introduce the norm.

MML Identifier: `CSSPACE2`.

The papers [18], [21], [5], [17], [10], [22], [3], [4], [20], [19], [13], [11], [12], [15], [2], [1], [14], [16], [6], [8], and [7] provide the notation and terminology for this paper.

1. Hilbert Space of Complex Sequences

One can prove the following propositions:

(1) The carrier of Complexl2-Space = the set of l2-complex sequences and for every set x holds x is an element of Complexl2-Space iff x is a complex sequence and $|\mathrm{id}_{\mathrm{seq}}(x)|\,|\mathrm{id}_{\mathrm{seq}}(x)|$ is summable and for every set x holds x is an element of Complexl2-Space iff x is a complex sequence and $\mathrm{id}_{\mathrm{seq}}(x)\,\overline{\mathrm{id}_{\mathrm{seq}}(x)}$ is absolutely summable and $0_{\mathrm{Complexl2\text{-}Space}} = \mathrm{CZeroseq}$ and for every vector u of Complexl2-Space holds $u = \mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of Complexl2-Space holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$ and for every Complex r and for every vector u of Complexl2-Space holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$ and for every vector u of Complexl2-Space holds $-u = -\mathrm{id}_{\mathrm{seq}}(u)$ and $\mathrm{id}_{\mathrm{seq}}(-u) = -\mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of Complexl2-Space holds $u - v = \mathrm{id}_{\mathrm{seq}}(u) - \mathrm{id}_{\mathrm{seq}}(v)$ and for all vectors v, w of Complexl2-Space holds $|\mathrm{id}_{\mathrm{seq}}(v)|\,|\mathrm{id}_{\mathrm{seq}}(w)|$ is summable and for all vectors v, w of Complexl2-Space holds $(v|w) = \sum(\mathrm{id}_{\mathrm{seq}}(v)\,\overline{\mathrm{id}_{\mathrm{seq}}(w)})$.

(2) Let x, y, z be points of Complexl2-Space and a be a Complex. Then $(x|x) = 0$ iff $x = 0_{\mathrm{Complexl2\text{-}Space}}$ and $\Re((x|x)) \geqslant 0$ and $\Im((x|x)) = 0$ and $(x|y) = \overline{(y|x)}$ and $((x + y)|z) = (x|z) + (y|z)$ and $((a \cdot x)|y) = a \cdot (x|y)$.

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One can verify that Complexl2-Space is complex unitary space-like.

Next we state the proposition

(3) For every sequence v_1 of Complexl2-Space such that v_1 is Cauchy holds v_1 is convergent.

Let us mention that Complexl2-Space is Hilbert.

2. Some Corollaries of Complex Sequences

Next we state a number of propositions:

(4) For all Complexes z_1, z_2 such that $\Re(z_1) \cdot \Im(z_2) = \Re(z_2) \cdot \Im(z_1)$ and $\Re(z_1) \cdot \Re(z_2) + \Im(z_1) \cdot \Im(z_2) \geqslant 0$ holds $|z_1 + z_2| = |z_1| + |z_2|$.

(5) For all Complexes x, y holds $2 \cdot |x \cdot y| \leqslant |x|^{\mathbf{2}} + |y|^{\mathbf{2}}$.

(6) For all Complexes x, y holds $|x+y| \cdot |x+y| \leqslant 2 \cdot |x| \cdot |x| + 2 \cdot |y| \cdot |y|$ and $|x| \cdot |x| \leqslant 2 \cdot |x-y| \cdot |x-y| + 2 \cdot |y| \cdot |y|$.

(7) For every complex sequence s_1 holds $s_1 = \overline{\overline{s_1}}$.

(8) For every complex sequence s_1 holds $(\sum_{\alpha=0}^{\kappa} \overline{s_1}(\alpha))_{\kappa \in \mathbb{N}} = \overline{(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}}$.

(9) Let s_1 be a complex sequence and n be a natural number. Suppose that for every natural number i holds $\Re(s_1)(i) \geqslant 0$ and $\Im(s_1)(i) = 0$. Then $|(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}|(n) = (\sum_{\alpha=0}^{\kappa} |s_1|(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(10) For every complex sequence s_1 such that s_1 is summable holds $\sum \overline{s_1} = \overline{\sum s_1}$.

(11) For every complex sequence s_1 such that s_1 is absolutely summable holds $|\sum s_1| \leqslant \sum |s_1|$.

(12) Let s_1 be a complex sequence. Suppose s_1 is summable and for every natural number n holds $\Re(s_1)(n) \geqslant 0$ and $\Im(s_1)(n) = 0$. Then $|\sum s_1| = \sum |s_1|$.

(13) For every complex sequence s_1 and for every natural number n holds $\Re(s_1\, \overline{s_1})(n) \geqslant 0$ and $\Im(s_1\, \overline{s_1})(n) = 0$.

(14) Let s_1 be a complex sequence. Suppose s_1 is absolutely summable and $\sum |s_1| = 0$. Let n be a natural number. Then $s_1(n) = 0_{\mathbb{C}}$.

(15) For every complex sequence s_1 holds $|s_1| = |\overline{s_1}|$.

(16) Let c be a Complex and s_1 be a complex sequence. Suppose s_1 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number m holds $r_1(m) = |s_1(m) - c| \cdot |s_1(m) - c|$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c| \cdot |\lim s_1 - c|$.

(17) Let c be a Complex, s_2 be a sequence of real numbers, and s_1 be a complex sequence. Suppose s_1 is convergent and s_2 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number i

holds $r_1(i) = |s_1(i) - c| \cdot |s_1(i) - c| + s_2(i)$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c| \cdot |\lim s_1 - c| + \lim s_2$.

(18) Let c be a Complex and s_1 be a complex sequence. Suppose s_1 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number m holds $r_1(m) = |s_1(m) - c| \cdot |s_1(m) - c|$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c| \cdot |\lim s_1 - c|$.

(19) Let c be a Complex, s_2 be a sequence of real numbers, and s_1 be a complex sequence. Suppose s_1 is convergent and s_2 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number i holds $r_1(i) = |s_1(i) - c| \cdot |s_1(i) - c| + s_2(i)$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c| \cdot |\lim s_1 - c| + \lim s_2$.

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Banach Space of Absolute Summable Complex Sequences

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Summary. An extension of [16]. As the example of complex norm spaces, I introduced the arithmetic addition and multiplication in the set of absolute summable complex sequences and also introduced the norm.

MML Identifier: `CSSPACE3`.

The terminology and notation used in this paper are introduced in the following articles: [18], [20], [6], [2], [17], [9], [21], [4], [5], [19], [13], [11], [10], [14], [3], [1], [12], [15], [7], and [8].

1. Complex-L1-Space: The Space of Absolute Summable Complex Sequences

The subset the set of l1-complex sequences of the linear space of complex sequences is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ the set of l1-complex sequences if and only if $x \in$ the set of complex sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is absolutely summable.

The following proposition is true

(1) Let c be a Complex, s_1 be a complex sequence, and r_1 be a sequence of real numbers. Suppose s_1 is convergent and for every natural number i holds $r_1(i) = |s_1(i) - c|$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c|$.

Let us note that the set of l1-complex sequences is non empty.

Let us observe that the set of l1-complex sequences is linearly closed.

Next we state the proposition

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(2) ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences)⟩ is a subspace of the linear space of complex sequences.

Let us note that ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences)⟩ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

We now state the proposition

(3) ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences)⟩ is a complex linear space.

The function cl_norm from the set of l1-complex sequences into $\mathbb{R}$ is defined as follows:

(Def. 2) For every set x such that $x \in$ the set of l1-complex sequences holds $\mathrm{cl\_norm}(x) = \sum |\mathrm{id}_{\mathrm{seq}}(x)|$.

Let X be a non empty set, let Z be an element of X, let A be a binary operation on X, let M be a function from $[:\mathbb{C}, X:]$ into X, and let N be a function from X into $\mathbb{R}$. Note that $\langle X, Z, A, M, N\rangle$ is non empty.

We now state four propositions:

(4) Let l be a complex normed space structure. Suppose ⟨the carrier of l, the zero of l, the addition of l, the external multiplication of l⟩ is a complex linear space. Then l is a complex linear space.

(5) Let c_1 be a complex sequence. Suppose that for every natural number n holds $c_1(n) = 0_{\mathbb{C}}$. Then c_1 is absolutely summable and $\sum |c_1| = 0$.

(6) Let c_1 be a complex sequence. Suppose c_1 is absolutely summable and $\sum |c_1| = 0$. Let n be a natural number. Then $c_1(n) = 0_{\mathbb{C}}$.

(7) ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences), cl_norm⟩ is a complex linear space.

The non empty complex normed space structure Complex-l1-Space is defined by the condition (Def. 3).

(Def. 3) Complex-l1-Space = ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences), cl_norm⟩.

2. Complex-L1-Space is Banach

One can prove the following propositions:

(8) The carrier of Complex-l1-Space = the set of l1-complex sequences and for every set x holds x is a vector of Complex-l1-Space iff x is a complex sequence and $\mathrm{id_{seq}}(x)$ is absolutely summable and $0_{\text{Complex-l1-Space}} =$ CZeroseq and for every vector u of Complex-l1-Space holds $u = \mathrm{id_{seq}}(u)$ and for all vectors u, v of Complex-l1-Space holds $u+v = \mathrm{id_{seq}}(u)+\mathrm{id_{seq}}(v)$ and for every Complex p and for every vector u of Complex-l1-Space holds $p \cdot u = p\ \mathrm{id_{seq}}(u)$ and for every vector u of Complex-l1-Space holds $-u = -\mathrm{id_{seq}}(u)$ and $\mathrm{id_{seq}}(-u) = -\mathrm{id_{seq}}(u)$ and for all vectors u, v of Complex-l1-Space holds $u - v = \mathrm{id_{seq}}(u) - \mathrm{id_{seq}}(v)$ and for every vector v of Complex-l1-Space holds $\mathrm{id_{seq}}(v)$ is absolutely summable and for every vector v of Complex-l1-Space holds $\|v\| = \sum |\mathrm{id_{seq}}(v)|$.

(9) Let x, y be points of Complex-l1-Space and p be a Complex. Then $\|x\| = 0$ iff $x = 0_{\text{Complex-l1-Space}}$ and $0 \leqslant \|x\|$ and $\|x + y\| \leqslant \|x\| + \|y\|$ and $\|p \cdot x\| = |p| \cdot \|x\|$.

Let us observe that Complex-l1-Space is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, and right complementable.

Let X be a non empty complex normed space structure and let x, y be points of X. The functor $\rho(x, y)$ yielding a real number is defined as follows:

(Def. 4) $\rho(x, y) = \|x - y\|$.

Let C_1 be a non empty complex normed space structure and let s_2 be a sequence of C_1. We say that s_2 is CCauchy if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let r_2 be a real number. Suppose $r_2 > 0$. Then there exists a natural number k_1 such that for all natural numbers n_1, m_1 if $n_1 \geqslant k_1$ and $m_1 \geqslant k_1$, then $\rho(s_2(n_1), s_2(m_1)) < r_2$.

We introduce s_1 is Cauchy sequence by norm as a synonym of s_2 is CCauchy.

In the sequel N_1 is a non empty complex normed space and s_1 is a sequence of N_1.

One can prove the following propositions:

(10) s_1 is Cauchy sequence by norm if and only if for every real number r such that $r > 0$ there exists a natural number k such that for all natural numbers n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|s_1(n) - s_1(m)\| < r$.

(11) For every sequence v_1 of Complex-l1-Space such that v_1 is Cauchy sequence by norm holds v_1 is convergent.

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The Taylor Expansions

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Summary. In this article, some classic theorems of calculus are described. The Taylor expansions and the logarithmic differentiation, etc. are included here.

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The terminology and notation used in this paper have been introduced in the following articles: [22], [24], [25], [4], [6], [9], [5], [11], [20], [18], [3], [8], [2], [21], [7], [1], [23], [14], [12], [10], [17], [19], [13], [15], [16], and [26].

1. The Logarithmic Differentiation Method

For simplicity, we use the following convention: n denotes a natural number, i denotes an integer, p, x, x_0, y denote real numbers, q denotes a rational number, and f denotes a partial function from $\mathbb{R}$ to $\mathbb{R}$.

Let q be an integer. The functor $\frac{q}{\mathbb{Z}}$ yields a function from $\mathbb{R}$ into $\mathbb{R}$ and is defined as follows:

(Def. 1) For every real number x holds $(\frac{q}{\mathbb{Z}})(x) = x^{q}_{\mathbb{Z}}$.

Next we state a number of propositions:

(1) For all natural numbers m, n holds $x^{n+m}_{\mathbb{Z}} = (x^{n}_{\mathbb{Z}}) \cdot x^{m}_{\mathbb{Z}}$.

(2) $\frac{n}{\mathbb{Z}}$ is differentiable in x and $(\frac{n}{\mathbb{Z}})'(x) = n \cdot x^{n-1}_{\mathbb{Z}}$.

(3) If f is differentiable in x_0, then $(\frac{n}{\mathbb{Z}}) \cdot f$ is differentiable in x_0 and $((\frac{n}{\mathbb{Z}}) \cdot f)'(x_0) = n \cdot f(x_0)^{n-1}_{\mathbb{Z}} \cdot f'(x_0)$.

(4) $\exp(-x) = \frac{1}{\exp x}$.

(5) $(\exp x)^{\frac{1}{i}}_{\mathbb{R}} = \exp(\frac{x}{i})$.

(6) For all integers m, n holds $(\exp x)^{\frac{m}{n}}_{\mathbb{R}} = \exp(\frac{m}{n} \cdot x)$.

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(7) $(\exp x)^{q}_{\mathbb{Q}} = \exp(q \cdot x)$.

(8) $(\exp x)^{p}_{\mathbb{R}} = \exp(p \cdot x)$.

(9) $(\exp 1)^{x}_{\mathbb{R}} = \exp x$ and $(\exp 1)^{x} = \exp x$ and $e^{x} = \exp x$ and $e^{x}_{\mathbb{R}} = \exp x$.

(10) $\exp(1)^{x}_{\mathbb{R}} = \exp(x)$ and $\exp(1)^{x} = \exp(x)$ and $e^{x} = \exp(x)$ and $e^{x}_{\mathbb{R}} = \exp(x)$.

(11) $e \geqslant 2$.

(12) $\log_{e} \exp x = x$.

(13) $\log_{e} \exp(x) = x$.

(14) If $y > 0$, then $\exp \log_{e} y = y$.

(15) If $y > 0$, then $\exp(\log_{e} y) = y$.

(16) exp is one-to-one and exp is differentiable on $\mathbb{R}$ and exp is differentiable on $\Omega_{\mathbb{R}}$ and for every real number x holds $\exp'(x) = \exp(x)$ and for every real number x holds $0 < \exp'(x)$ and $\mathrm{dom} \exp = \mathbb{R}$ and $\mathrm{dom} \exp = \Omega_{\mathbb{R}}$ and $\mathrm{rng} \exp =]0, +\infty[$.

Let us note that exp is one-to-one.

We now state the proposition

(17) $\exp^{-1}$ is differentiable on $\mathrm{dom}(\exp^{-1})$ and for every real number x such that $x \in \mathrm{dom}(\exp^{-1})$ holds $(\exp^{-1})'(x) = \frac{1}{x}$.

Let us mention that $]0, +\infty[$ is non empty.

Let a be a real number. The functor $\log\_(a)$ yields a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 2) $\mathrm{dom} \log\_(a) =]0, +\infty[$ and for every element d of $]0, +\infty[$ holds $(\log\_(a))(d) = \log_{a} d$.

One can prove the following three propositions:

(18) $\log\_(e) = \exp^{-1}$ and $\log\_(e)$ is one-to-one and $\mathrm{dom} \log\_(e) =]0, +\infty[$ and $\mathrm{rng} \log\_(e) = \mathbb{R}$ and $\log\_(e)$ is differentiable on $]0, +\infty[$ and for every real number x such that $x > 0$ holds $\log\_(e)$ is differentiable in x and for every element x of $]0, +\infty[$ holds $(\log\_(e))'(x) = \frac{1}{x}$ and for every element x of $]0, +\infty[$ holds $0 < (\log\_(e))'(x)$.

(19) If f is differentiable in x_0, then $\exp \cdot f$ is differentiable in x_0 and $(\exp \cdot f)'(x_0) = \exp(f(x_0)) \cdot f'(x_0)$.

(20) If f is differentiable in x_0 and $f(x_0) > 0$, then $\log\_(e) \cdot f$ is differentiable in x_0 and $(\log\_(e) \cdot f)'(x_0) = \frac{f'(x_0)}{f(x_0)}$.

Let p be a real number. The functor $^{p}_{\mathbb{R}}$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 3) $\mathrm{dom}(^{p}_{\mathbb{R}}) =]0, +\infty[$ and for every element d of $]0, +\infty[$ holds $(^{p}_{\mathbb{R}})(d) = d^{p}_{\mathbb{R}}$.

We now state two propositions:

(21) If $x > 0$, then $^{p}_{\mathbb{R}}$ is differentiable in x and $(^{p}_{\mathbb{R}})'(x) = p \cdot x^{p-1}_{\mathbb{R}}$.

(22) If f is differentiable in x_0 and $f(x_0) > 0$, then $\binom{p}{\mathbb{R}} \cdot f$ is differentiable in x_0 and $(\binom{p}{\mathbb{R}} \cdot f)'(x_0) = p \cdot f(x_0)_{\mathbb{R}}^{p-1} \cdot f'(x_0)$.

2. The Taylor Expansions

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let Z be a subset of $\mathbb{R}$. The functor $f'(Z)$ yields a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ and is defined by:

(Def. 4) $f'(Z)(0) = f{\restriction}Z$ and for every natural number i holds $f'(Z)(i+1) = f'(Z)(i)'_{\restriction Z}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, let n be a natural number, and let Z be a subset of $\mathbb{R}$. We say that f is differentiable n times on Z if and only if:

(Def. 5) For every natural number i such that $i \leqslant n-1$ holds $f'(Z)(i)$ is differentiable on Z.

The following proposition is true

(23) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and n be a natural number. Suppose f is differentiable n times on Z. Let m be a natural number. If $m \leqslant n$, then f is differentiable m times on Z.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, let Z be a subset of $\mathbb{R}$, and let a, b be real numbers. The functor $\mathrm{Taylor}(f, Z, a, b)$ yields a sequence of real numbers and is defined as follows:

(Def. 6) For every natural number n holds $(\mathrm{Taylor}(f, Z, a, b))(n) = \frac{f'(Z)(n)(a) \cdot (b-a)^n}{n!}$.

The following propositions are true:

(24) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and n be a natural number. Suppose f is differentiable n times on Z. Let a, b be real numbers. If $a < b$ and $]a, b[\subseteq Z$, then $f'(Z)(n){\restriction}]a, b[= f'(]a, b[)(n)$.

(25) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and Z be a subset of $\mathbb{R}$. Suppose f is differentiable n times on Z. Let a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq Z$ and $f'(Z)(n)$ is continuous on $[a, b]$ and f is differentiable $n+1$ times on $]a, b[$. Let l be a real number and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\mathrm{dom}\, g = \mathbb{R}$ and for every real number x holds $g(x) = f(b) - (\sum_{\alpha=0}^{\kappa} (\mathrm{Taylor}(f, Z, x, b))(\alpha))_{\kappa \in \mathbb{N}}(n) - \frac{l \cdot (b-x)^{n+1}}{(n+1)!}$ and $f(b) - (\sum_{\alpha=0}^{\kappa} (\mathrm{Taylor}(f, Z, a, b))(\alpha))_{\kappa \in \mathbb{N}}(n) - \frac{l \cdot (b-a)^{n+1}}{(n+1)!} = 0$. Then

(i) g is differentiable on $]a, b[$,

(ii) $g(a) = 0$,

(iii) $g(b) = 0$,

(iv) g is continuous on $[a, b]$, and

(v) for every real number x such that $x \in\ ]a, b[$ holds $g'(x) = -\frac{f'(]a,b[)(n+1)(x) \cdot (b-x)^n}{n!} + \frac{l \cdot (b-x)^n}{n!}$.

(26) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and b, l be real numbers. Then there exists a function g from $\mathbb{R}$ into $\mathbb{R}$ such that for every real number x holds $g(x) = f(b) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f, Z, x, b))(\alpha))_{\kappa\in\mathbb{N}}(n) - \frac{l\cdot(b-x)^{n+1}}{(n+1)!}$.

(27) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and Z be a subset of $\mathbb{R}$. Suppose f is differentiable n times on Z. Let a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq Z$ and $f'(Z)(n)$ is continuous on $[a, b]$ and f is differentiable $n+1$ times on $]a, b[$. Then there exists a real number c such that $c \in\ ]a, b[$ and $f(b) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f, Z, a, b))(\alpha))_{\kappa\in\mathbb{N}}(n) + \frac{f'(]a,b[)(n+1)(c)\cdot(b-a)^{n+1}}{(n+1)!}$.

(28) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and Z be a subset of $\mathbb{R}$. Suppose f is differentiable n times on Z. Let a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq Z$ and $f'(Z)(n)$ is continuous on $[a, b]$ and f is differentiable $n + 1$ times on $]a, b[$. Let l be a real number and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\mathrm{dom}\, g = \mathbb{R}$ and for every real number x holds $g(x) = f(a) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f, Z, x, a))(\alpha))_{\kappa\in\mathbb{N}}(n) - \frac{l\cdot(a-x)^{n+1}}{(n+1)!}$ and $f(a) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f, Z, b, a))(\alpha))_{\kappa\in\mathbb{N}}(n) - \frac{l\cdot(a-b)^{n+1}}{(n+1)!} = 0$. Then

(i) g is differentiable on $]a, b[$,

(ii) $g(b) = 0$,

(iii) $g(a) = 0$,

(iv) g is continuous on $[a, b]$, and

(v) for every real number x such that $x \in\ ]a, b[$ holds $g'(x) = -\frac{f'(]a,b[)(n+1)(x)\cdot(a-x)^{n}}{n!} + \frac{l\cdot(a-x)^{n}}{n!}$.

(29) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and Z be a subset of $\mathbb{R}$. Suppose f is differentiable n times on Z. Let a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq Z$ and $f'(Z)(n)$ is continuous on $[a, b]$ and f is differentiable $n+1$ times on $]a, b[$. Then there exists a real number c such that $c \in\ ]a, b[$ and $f(a) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f, Z, b, a))(\alpha))_{\kappa\in\mathbb{N}}(n) + \frac{f'(]a,b[)(n+1)(c)\cdot(a-b)^{n+1}}{(n+1)!}$.

(30) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and Z_1 be an open subset of $\mathbb{R}$. Suppose $Z_1 \subseteq Z$. Let n be a natural number. If f is differentiable n times on Z, then $f'(Z)(n){\restriction}Z_1 = f'(Z_1)(n)$.

(31) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and Z_1 be an open subset of $\mathbb{R}$. Suppose $Z_1 \subseteq Z$. Let n be a natural number. Suppose f is differentiable $n + 1$ times on Z. Then f is differentiable $n + 1$ times on Z_1.

(32) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and x be a real number. If $x \in Z$, then for every natural number n holds $f(x) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f, Z, x, x))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(33) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and x_0, r be real numbers. Suppose $0 < r$ and f is differentiable $n+1$ times on $]x_0 - r, x_0 + r[$. Let x be a real number. Suppose $x \in]x_0 - r, x_0 + r[$. Then there exists a real number s such that $0 < s$ and $s < 1$ and $f(x) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f,]x_0 - r, x_0 + r[, x_0, x))(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{f'(]x_0 - r, x_0 + r[)(n+1)(x_0 + s \cdot (x - x_0)) \cdot (x - x_0)^{n+1}}{(n+1)!}$.

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Complex Banach Space of Bounded Linear Operators

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Summary. An extension of [19]. In this article, the basic properties of complex linear spaces which are defined by the set of all complex linear operators from one complex linear space to another are described. Finally, a complex Banach space is introduced. This is defined by the set of all bounded complex linear operators, like in [19].

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The articles [24], [6], [26], [27], [4], [5], [17], [22], [21], [2], [1], [20], [11], [7], [25], [23], [18], [15], [13], [14], [12], [16], [3], [9], [10], [8], and [19] provide the terminology and notation for this paper.

1. Complex Vector Space of Operators

Let X be a set, let Y be a non empty set, let F be a function from $[\!:\mathbb{C}, Y:\!]$ into Y, let c be a complex number, and let f be a function from X into Y. Then $F^{\circ}(c, f)$ is an element of Y^X.

We now state the proposition

(1) Let X be a non empty set and Y be a complex linear space. Then there exists a function M_1 from $[\!:\mathbb{C}, (\text{the carrier of } Y)^X:\!]$ into (the carrier of $Y)^X$ such that for every Complex c and for every element f of (the carrier of $Y)^X$ and for every element s of X holds $M_1(\langle c, f\rangle)(s) = c \cdot f(s)$.

Let X be a non empty set and let Y be a complex linear space. The functor FuncExtMult(X, Y) yields a function from $[\!:\mathbb{C}, (\text{the carrier of } Y)^X:\!]$ into (the carrier of $Y)^X$ and is defined by the condition (Def. 1).

(Def. 1) Let c be a Complex, f be an element of (the carrier of $Y)^X$, and x be an element of X. Then $(\text{FuncExtMult}(X, Y))(\langle c, f\rangle)(x) = c \cdot f(x)$.

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We follow the rules: X is a non empty set, Y is a complex linear space, and f, g, h are elements of (the carrier of $Y)^X$.

We now state the proposition

(2) For every element x of X holds $(\mathrm{FuncZero}(X,Y))(x) = 0_Y$.

In the sequel a, b are Complexes.

Next we state several propositions:

(3) $h = (\mathrm{FuncExtMult}(X,Y))(\langle a, f\rangle)$ iff for every element x of X holds $h(x) = a \cdot f(x)$.

(4) $(\mathrm{FuncAdd}(X,Y))(f, g) = (\mathrm{FuncAdd}(X,Y))(g, f)$.

(5) $(\mathrm{FuncAdd}(X,Y))(f, (\mathrm{FuncAdd}(X,Y))(g, h)) = (\mathrm{FuncAdd}(X,Y))((\mathrm{FuncAdd}(X,Y))(f, g), h)$.

(6) $(\mathrm{FuncAdd}(X,Y))(\mathrm{FuncZero}(X,Y), f) = f$.

(7) $(\mathrm{FuncAdd}(X,Y))(f, (\mathrm{FuncExtMult}(X,Y))(\langle -1_{\mathbb{C}}, f\rangle)) = \mathrm{FuncZero}(X,Y)$.

(8) $(\mathrm{FuncExtMult}(X,Y))(\langle 1_{\mathbb{C}}, f\rangle) = f$.

(9) $(\mathrm{FuncExtMult}(X,Y))(\langle a, (\mathrm{FuncExtMult}(X,Y))(\langle b, f\rangle)\rangle) = (\mathrm{FuncExtMult}(X,Y))(\langle a \cdot b, f\rangle)$.

(10) $(\mathrm{FuncAdd}(X,Y))((\mathrm{FuncExtMult}(X,Y))(\langle a, f\rangle), (\mathrm{FuncExtMult}(X,Y))(\langle b, f\rangle)) = (\mathrm{FuncExtMult}(X,Y))(\langle a + b, f\rangle)$.

(11) $\langle$(the carrier of $Y)^X, \mathrm{FuncZero}(X,Y), \mathrm{FuncAdd}(X,Y), \mathrm{FuncExtMult}(X,Y)\rangle$ is a complex linear space.

Let X be a non empty set and let Y be a complex linear space. The functor ComplexVectSpace(X,Y) yielding a complex linear space is defined as follows:

(Def. 2) $\mathrm{ComplexVectSpace}(X,Y) = \langle$(the carrier of $Y)^X, \mathrm{FuncZero}(X,Y), \mathrm{FuncAdd}(X,Y), \mathrm{FuncExtMult}(X,Y)\rangle$.

Let X be a non empty set and let Y be a complex linear space. Observe that ComplexVectSpace(X,Y) is strict.

Let X be a non empty set and let Y be a complex linear space. Observe that every vector of ComplexVectSpace(X,Y) is function-like and relation-like.

Let X be a non empty set, let Y be a complex linear space, let f be a vector of ComplexVectSpace(X,Y), and let x be an element of X. Then $f(x)$ is a vector of Y.

We now state three propositions:

(12) Let X be a non empty set, Y be a complex linear space, and f, g, h be vectors of ComplexVectSpace(X,Y). Then $h = f + g$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(13) Let X be a non empty set, Y be a complex linear space, f, h be vectors of ComplexVectSpace(X,Y), and c be a Complex. Then $h = c \cdot f$ if and only if for every element x of X holds $h(x) = c \cdot f(x)$.

(14) For every non empty set X and for every complex linear space Y holds $0_{\mathrm{ComplexVectSpace}(X,Y)} = X \longmapsto 0_Y$.

2. Complex Vector Space of Linear Operators

Let X be a non empty CLS structure, let Y be a non empty loop structure, and let I_1 be a function from X into Y. We say that I_1 is additive if and only if:

(Def. 3) For all vectors x, y of X holds $I_1(x + y) = I_1(x) + I_1(y)$.

Let X, Y be non empty CLS structures and let I_1 be a function from X into Y. We say that I_1 is homogeneous if and only if:

(Def. 4) For every vector x of X and for every Complex r holds $I_1(r \cdot x) = r \cdot I_1(x)$.

Let X be a non empty CLS structure and let Y be a complex linear space. One can verify that there exists a function from X into Y which is additive and homogeneous.

Let X, Y be complex linear spaces. A linear operator from X into Y is an additive homogeneous function from X into Y.

Let X, Y be complex linear spaces. The functor LinearOperators(X, Y) yielding a subset of ComplexVectSpace(the carrier of X, Y) is defined by:

(Def. 5) For every set x holds $x \in$ LinearOperators(X, Y) iff x is a linear operator from X into Y.

Let X, Y be complex linear spaces. Note that LinearOperators(X, Y) is non empty.

Next we state two propositions:

(15) For all complex linear spaces X, Y holds LinearOperators(X, Y) is linearly closed.

(16) Let X, Y be complex linear spaces. Then $\langle$LinearOperators(X, Y), Zero_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y)), Add_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y))$\rangle$ is a subspace of ComplexVectSpace(the carrier of X, Y).

Let X, Y be complex linear spaces. One can check that $\langle$LinearOperators(X, Y), Zero_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y)), Add_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y))$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

Next we state the proposition

(17) Let X, Y be complex linear spaces. Then $\langle$LinearOperators(X, Y), Zero_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y)),

Add_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y))⟩ is a complex linear space.

Let X, Y be complex linear spaces. The functor CVSpLinOps(X, Y) yielding a complex linear space is defined as follows:

(Def. 6) CVSpLinOps(X, Y) = ⟨LinearOperators(X, Y), Zero_(LinearOperators (X, Y), ComplexVectSpace(the carrier of X, Y)), Add_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y))⟩.

Let X, Y be complex linear spaces. Note that CVSpLinOps(X, Y) is strict.

Let X, Y be complex linear spaces. One can check that every element of CVSpLinOps(X, Y) is function-like and relation-like.

Let X, Y be complex linear spaces, let f be an element of CVSpLinOps(X, Y), and let v be a vector of X. Then $f(v)$ is a vector of Y.

Next we state four propositions:

(18) Let X, Y be complex linear spaces and f, g, h be vectors of CVSpLinOps(X, Y). Then $h = f + g$ if and only if for every vector x of X holds $h(x) = f(x) + g(x)$.

(19) Let X, Y be complex linear spaces, f, h be vectors of CVSpLinOps(X, Y), and c be a Complex. Then $h = c \cdot f$ if and only if for every vector x of X holds $h(x) = c \cdot f(x)$.

(20) For all complex linear spaces X, Y holds $0_{\mathrm{CVSpLinOps}(X,Y)} =$ (the carrier of X) $\longmapsto 0_Y$.

(21) For all complex linear spaces X, Y holds (the carrier of X) $\longmapsto 0_Y$ is a linear operator from X into Y.

3. Complex Normed Linear Space of Bounded Linear Operators

One can prove the following proposition

(22) Let X be a complex normed space, s_1 be a sequence of X, and g be a point of X. If s_1 is convergent and $\lim s_1 = g$, then $\|s_1\|$ is convergent and $\lim \|s_1\| = \|g\|$.

Let X, Y be complex normed spaces and let I_1 be a linear operator from X into Y. We say that I_1 is bounded if and only if:

(Def. 7) There exists a real number K such that $0 \leqslant K$ and for every vector x of X holds $\|I_1(x)\| \leqslant K \cdot \|x\|$.

We now state the proposition

(23) Let X, Y be complex normed spaces and f be a linear operator from X into Y. If for every vector x of X holds $f(x) = 0_Y$, then f is bounded.

Let X, Y be complex normed spaces. Observe that there exists a linear operator from X into Y which is bounded.

Let X, Y be complex normed spaces. The functor $\mathrm{BdLinOps}(X, Y)$ yielding a subset of $\mathrm{CVSpLinOps}(X, Y)$ is defined as follows:

(Def. 8) For every set x holds $x \in \mathrm{BdLinOps}(X, Y)$ iff x is a bounded linear operator from X into Y.

Let X, Y be complex normed spaces. One can check that $\mathrm{BdLinOps}(X, Y)$ is non empty.

One can prove the following two propositions:

(24) For all complex normed spaces X, Y holds $\mathrm{BdLinOps}(X, Y)$ is linearly closed.

(25) For all complex normed spaces X, Y holds $\langle \mathrm{BdLinOps}(X, Y), \mathrm{Zero\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y)), \mathrm{Add\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y)), \mathrm{Mult\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y))\rangle$ is a subspace of $\mathrm{CVSpLinOps}(X, Y)$.

Let X, Y be complex normed spaces. Observe that $\langle \mathrm{BdLinOps}(X, Y), \mathrm{Zero\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y)), \mathrm{Add\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y)), \mathrm{Mult\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y))\rangle$ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

Next we state the proposition

(26) For all complex normed spaces X, Y holds $\langle \mathrm{BdLinOps}(X, Y), \mathrm{Zero\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y)), \mathrm{Add\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y)), \mathrm{Mult\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y))\rangle$ is a complex linear space.

Let X, Y be complex normed spaces. The functor $\mathrm{CVSpBdLinOps}(X, Y)$ yielding a complex linear space is defined by:

(Def. 9) $\mathrm{CVSpBdLinOps}(X, Y) = \langle \mathrm{BdLinOps}(X, Y), \mathrm{Zero\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y)), \mathrm{Add\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y)), \mathrm{Mult\_}(\mathrm{BdLinOps}(X, Y), \mathrm{CVSpLinOps}(X, Y))\rangle$.

Let X, Y be complex normed spaces. One can check that $\mathrm{CVSpBdLinOps}(X, Y)$ is strict.

Let X, Y be complex normed spaces. Note that every element of $\mathrm{CVSpBdLinOps}(X, Y)$ is function-like and relation-like.

Let X, Y be complex normed spaces, let f be an element of $\mathrm{CVSpBdLinOps}(X, Y)$, and let v be a vector of X. Then $f(v)$ is a vector of Y.

One can prove the following propositions:

(27) Let X, Y be complex normed spaces and f, g, h be vectors of $\mathrm{CVSpBdLinOps}(X, Y)$. Then $h = f + g$ if and only if for every vector

x of X holds $h(x) = f(x) + g(x)$.

(28) Let X, Y be complex normed spaces, f, h be vectors of CVSpBdLinOps(X, Y), and c be a Complex. Then $h = c \cdot f$ if and only if for every vector x of X holds $h(x) = c \cdot f(x)$.

(29) For all complex normed spaces X, Y holds $0_{\mathrm{CVSpBdLinOps}(X,Y)} =$ (the carrier of X) $\longmapsto 0_Y$.

Let X, Y be complex normed spaces and let f be a set. Let us assume that $f \in$ BdLinOps(X, Y). The functor modetrans(f, X, Y) yields a bounded linear operator from X into Y and is defined as follows:

(Def. 10) modetrans$(f, X, Y) = f$.

Let X, Y be complex normed spaces and let u be a linear operator from X into Y. The functor PreNorms(u) yielding a non empty subset of $\mathbb{R}$ is defined as follows:

(Def. 11) PreNorms$(u) = \{\|u(t)\|; t$ ranges over vectors of X: $\|t\| \leqslant 1\}$.

We now state three propositions:

(30) Let X, Y be complex normed spaces and g be a bounded linear operator from X into Y. Then PreNorms(g) is non empty and upper bounded.

(31) Let X, Y be complex normed spaces and g be a linear operator from X into Y. Then g is bounded if and only if PreNorms(g) is upper bounded.

(32) Let X, Y be complex normed spaces. Then there exists a function N_1 from BdLinOps(X, Y) into $\mathbb{R}$ such that for every set f if $f \in$ BdLinOps(X, Y), then $N_1(f) = \sup$ PreNorms(modetrans(f, X, Y)).

Let X, Y be complex normed spaces. The functor BdLinOpsNorm(X, Y) yields a function from BdLinOps(X, Y) into $\mathbb{R}$ and is defined by:

(Def. 12) For every set x such that $x \in$ BdLinOps(X, Y) holds (BdLinOpsNorm(X, Y))$(x) = \sup$ PreNorms(modetrans(x, X, Y)).

We now state two propositions:

(33) For all complex normed spaces X, Y and for every bounded linear operator f from X into Y holds modetrans$(f, X, Y) = f$.

(34) For all complex normed spaces X, Y and for every bounded linear operator f from X into Y holds (BdLinOpsNorm(X, Y))$(f) =$ sup PreNorms(f).

Let X, Y be complex normed spaces. The functor CNSpBdLinOps(X, Y) yields a non empty complex normed space structure and is defined by:

(Def. 13) CNSpBdLinOps$(X, Y) = \langle$BdLinOps(X, Y), Zero_(BdLinOps(X, Y), CVSpLinOps(X, Y)), Add_(BdLinOps(X, Y), CVSpLinOps(X, Y)), Mult_(BdLinOps(X, Y), CVSpLinOps(X, Y)), BdLinOpsNorm$(X, Y)\rangle$.

The following four propositions are true:

(35) For all complex normed spaces X, Y holds (the carrier of X) $\longmapsto 0_Y = 0_{\mathrm{CNSpBdLinOps}(X,Y)}$.

(36) Let X, Y be complex normed spaces, f be a point of CNSpBdLinOps(X, Y), and g be a bounded linear operator from X into Y. If $g = f$, then for every vector t of X holds $\|g(t)\| \leqslant \|f\| \cdot \|t\|$.

(37) For all complex normed spaces X, Y and for every point f of CNSpBdLinOps(X, Y) holds $0 \leqslant \|f\|$.

(38) For all complex normed spaces X, Y and for every point f of CNSpBdLinOps(X, Y) such that $f = 0_{\mathrm{CNSpBdLinOps}(X,Y)}$ holds $0 = \|f\|$.

Let X, Y be complex normed spaces. One can check that every element of CNSpBdLinOps(X, Y) is function-like and relation-like.

Let X, Y be complex normed spaces, let f be an element of CNSpBdLinOps(X, Y), and let v be a vector of X. Then $f(v)$ is a vector of Y.

We now state several propositions:

(39) Let X, Y be complex normed spaces and f, g, h be points of CNSpBdLinOps(X, Y). Then $h = f + g$ if and only if for every vector x of X holds $h(x) = f(x) + g(x)$.

(40) Let X, Y be complex normed spaces, f, h be points of CNSpBdLinOps(X, Y), and c be a Complex. Then $h = c \cdot f$ if and only if for every vector x of X holds $h(x) = c \cdot f(x)$.

(41) Let X, Y be complex normed spaces, f, g be points of CNSpBdLinOps(X, Y), and c be a Complex. Then $\|f\| = 0$ iff $f = 0_{\mathrm{CNSpBdLinOps}(X,Y)}$ and $\|c \cdot f\| = |c| \cdot \|f\|$ and $\|f + g\| \leqslant \|f\| + \|g\|$.

(42) For all complex normed spaces X, Y holds CNSpBdLinOps(X, Y) is complex normed space-like.

(43) For all complex normed spaces X, Y holds CNSpBdLinOps(X, Y) is a complex normed space.

Let X, Y be complex normed spaces. Observe that CNSpBdLinOps(X, Y) is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following proposition

(44) Let X, Y be complex normed spaces and f, g, h be points of CNSpBdLinOps(X, Y). Then $h = f - g$ if and only if for every vector x of X holds $h(x) = f(x) - g(x)$.

4. Complex Banach Space of Bounded Linear Operators

Let X be a complex normed space. We say that X is complete if and only if:

(Def. 14) For every sequence s_1 of X such that s_1 is Cauchy sequence by norm holds s_1 is convergent.

Let us observe that there exists a complex normed space which is complete.

A complex Banach space is a complete complex normed space.

One can prove the following three propositions:

(45) Let X be a complex normed space and s_1 be a sequence of X. If s_1 is convergent, then $\|s_1\|$ is convergent and $\lim\|s_1\| = \|\lim s_1\|$.

(46) Let X, Y be complex normed spaces. Suppose Y is complete. Let s_1 be a sequence of CNSpBdLinOps(X, Y). If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(47) For every complex normed space X and for every complex Banach space Y holds CNSpBdLinOps(X, Y) is a complex Banach space.

Let X be a complex normed space and let Y be a complex Banach space. One can verify that CNSpBdLinOps(X, Y) is complete.

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Complex Banach Space of Bounded Complex Sequences

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Summary. An extension of [18]. In this article, we introduce two complex Banach spaces. One of them is the space of bounded complex sequences. The other one is the space of complex bounded functions, which is defined by the set of all complex bounded functions.

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The articles [21], [6], [23], [24], [17], [20], [2], [19], [12], [4], [5], [7], [22], [3], [1], [16], [15], [14], [10], [13], [11], [8], and [9] provide the terminology and notation for this paper.

1. Complex Banach Space of Bounded Complex Sequences

The subset the set of bounded complex sequences of the linear space of complex sequences is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ the set of bounded complex sequences if and only if $x \in$ the set of complex sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is bounded.

Let us note that the set of bounded complex sequences is non empty and the set of bounded complex sequences is linearly closed.

One can prove the following proposition

(1) $\langle$the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences), Mult_(the set of bounded complex sequences, the linear space of complex sequences)$\rangle$ is a subspace of the linear space of complex sequences.

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Let us mention that ⟨the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences),

Mult_(the set of bounded complex sequences, the linear space of complex sequences)⟩ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

The function Clinfty-norm from the set of bounded complex sequences into $\mathbb{R}$ is defined by:

(Def. 2) For every set x such that $x \in$ the set of bounded complex sequences holds Clinfty-norm$(x) = \sup \operatorname{rng} |\operatorname{id}_{\rm seq}(x)|$.

Next we state the proposition

(2) For every complex sequence s_1 holds s_1 is bounded and $\sup \operatorname{rng} |s_1| = 0$ iff for every natural number n holds $s_1(n) = 0_{\mathbb{C}}$.

One can check that ⟨the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences),

Mult_(the set of bounded complex sequences, the linear space of complex sequences), Clinfty-norm⟩ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

The non empty complex normed space structure Clinfty-Space is defined by the condition (Def. 3).

(Def. 3) Clinfty-Space = ⟨the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences), Mult_(the set of bounded complex sequences, the linear space of complex sequences), Clinfty-norm⟩.

Next we state two propositions:

(3) The carrier of Clinfty-Space = the set of bounded complex sequences and for every set x holds x is a vector of Clinfty-Space iff x is a complex sequence and $\operatorname{id}_{\rm seq}(x)$ is bounded and $0_{\rm Clinfty\text{-}Space}$ = CZeroseq and for every vector u of Clinfty-Space holds $u = \operatorname{id}_{\rm seq}(u)$ and for all vectors u, v of Clinfty-Space holds $u + v = \operatorname{id}_{\rm seq}(u) + \operatorname{id}_{\rm seq}(v)$ and for every Complex c and for every vector u of Clinfty-Space holds $c \cdot u = c \operatorname{id}_{\rm seq}(u)$ and for every vector u of Clinfty-Space holds $-u = -\operatorname{id}_{\rm seq}(u)$ and $\operatorname{id}_{\rm seq}(-u) = -\operatorname{id}_{\rm seq}(u)$ and for all vectors u, v of Clinfty-Space holds $u - v = \operatorname{id}_{\rm seq}(u) - \operatorname{id}_{\rm seq}(v)$ and for every vector v of Clinfty-Space holds $\operatorname{id}_{\rm seq}(v)$ is bounded and for every vector v of Clinfty-Space holds $\|v\| = \sup \operatorname{rng} |\operatorname{id}_{\rm seq}(v)|$.

(4) Let x, y be points of Clinfty-Space and c be a Complex. Then $\|x\| = 0$ iff $x = 0_{\rm Clinfty\text{-}Space}$ and $0 \leqslant \|x\|$ and $\|x+y\| \leqslant \|x\| + \|y\|$ and $\|c \cdot x\| = |c| \cdot \|x\|$.

Let us note that Clinfty-Space is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, and right complementable.

We now state two propositions:

(5) For every sequence v_1 of Clinfty-Space such that v_1 is Cauchy sequence by norm holds v_1 is convergent.

(6) Clinfty-Space is a complex Banach space.

2. Another Example of Complex Banach Space

Let X be a non empty set, let Y be a complex normed space, and let I_1 be a function from X into the carrier of Y. We say that I_1 is bounded if and only if:

(Def. 4) There exists a real number K such that $0 \leqslant K$ and for every element x of X holds $\|I_1(x)\| \leqslant K$.

The following proposition is true

(7) Let X be a non empty set, Y be a complex normed space, and f be a function from X into the carrier of Y. If for every element x of X holds $f(x) = 0_Y$, then f is bounded.

Let X be a non empty set and let Y be a complex normed space. One can check that there exists a function from X into the carrier of Y which is bounded.

Let X be a non empty set and let Y be a complex normed space. The functor $\mathrm{CBdFuncs}(X, Y)$ yields a subset of $\mathrm{ComplexVectSpace}(X, Y)$ and is defined by:

(Def. 5) For every set x holds $x \in \mathrm{CBdFuncs}(X, Y)$ iff x is a bounded function from X into the carrier of Y.

Let X be a non empty set and let Y be a complex normed space. Note that $\mathrm{CBdFuncs}(X, Y)$ is non empty.

One can prove the following propositions:

(8) For every non empty set X and for every complex normed space Y holds $\mathrm{CBdFuncs}(X, Y)$ is linearly closed.

(9) Let X be a non empty set and Y be a complex normed space. Then $\langle \mathrm{CBdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y)), \mathrm{Add\_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y))\rangle$ is a subspace of $\mathrm{ComplexVectSpace}(X, Y)$.

Let X be a non empty set and let Y be a complex normed space. Note that $\langle \mathrm{CBdFuncs}(X, Y), \mathrm{Zero\_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y)), \mathrm{Add\_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y)), \mathrm{Mult\_}(\mathrm{CBdFuncs}(X, Y), \mathrm{ComplexVectSpace}(X, Y))\rangle$ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

We now state the proposition

(10) Let X be a non empty set and Y be a complex normed space. Then $\langle \mathrm{CBdFuncs}(X,Y), \mathrm{Zero\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Add\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Mult\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y))\rangle$ is a complex linear space.

Let X be a non empty set and let Y be a complex normed space. The set of bounded complex sequences from X into Y yielding a complex linear space is defined by:

(Def. 6) The set of bounded complex sequences from X into Y = $\langle \mathrm{CBdFuncs}(X,Y), \mathrm{Zero\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Add\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Mult\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y))\rangle$.

Let X be a non empty set and let Y be a complex normed space. One can verify that the set of bounded complex sequences from X into Y is strict.

The following three propositions are true:

(11) Let X be a non empty set, Y be a complex normed space, f, g, h be vectors of the set of bounded complex sequences from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(12) Let X be a non empty set, Y be a complex normed space, f, h be vectors of the set of bounded complex sequences from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $h' = h$. Let c be a Complex. Then $h = c \cdot f$ if and only if for every element x of X holds $h'(x) = c \cdot f'(x)$.

(13) Let X be a non empty set and Y be a complex normed space. Then $0_{\text{the set of bounded complex sequences from } X \text{ into } Y} = X \longmapsto 0_Y$.

Let X be a non empty set, let Y be a complex normed space, and let f be a set. Let us assume that $f \in \mathrm{CBdFuncs}(X,Y)$. The functor $\mathrm{modetrans}(f,X,Y)$ yields a bounded function from X into the carrier of Y and is defined by:

(Def. 7) $\mathrm{modetrans}(f,X,Y) = f$.

Let X be a non empty set, let Y be a complex normed space, and let u be a function from X into the carrier of Y. The functor $\mathrm{PreNorms}(u)$ yielding a non empty subset of $\mathbb{R}$ is defined by:

(Def. 8) $\mathrm{PreNorms}(u) = \{\|u(t)\| : t$ ranges over elements of $X\}$.

We now state three propositions:

(14) Let X be a non empty set, Y be a complex normed space, and g be a bounded function from X into the carrier of Y. Then $\mathrm{PreNorms}(g)$ is non empty and upper bounded.

(15) Let X be a non empty set, Y be a complex normed space, and g be a function from X into the carrier of Y. Then g is bounded if and only if

PreNorms(g) is upper bounded.

(16) Let X be a non empty set and Y be a complex normed space. Then there exists a function N_1 from CBdFuncs(X, Y) into $\mathbb{R}$ such that for every set f if $f \in$ CBdFuncs(X, Y), then $N_1(f) = \sup$ PreNorms(modetrans(f, X, Y)).

Let X be a non empty set and let Y be a complex normed space. The functor CBdFuncsNorm(X, Y) yielding a function from CBdFuncs(X, Y) into $\mathbb{R}$ is defined by:

(Def. 9) For every set x such that $x \in$ CBdFuncs(X, Y) holds CBdFuncsNorm(X, Y)(x) $= \sup$ PreNorms(modetrans(x, X, Y)).

One can prove the following propositions:

(17) Let X be a non empty set, Y be a complex normed space, and f be a bounded function from X into the carrier of Y. Then modetrans(f, X, Y) $= f$.

(18) Let X be a non empty set, Y be a complex normed space, and f be a bounded function from X into the carrier of Y. Then CBdFuncsNorm(X, Y)(f) $= \sup$ PreNorms(f).

Let X be a non empty set and let Y be a complex normed space. The complex normed space of bounded functions from X into Y yields a non empty complex normed space structure and is defined by:

(Def. 10) The complex normed space of bounded functions from X into Y $=$ $\langle$CBdFuncs(X, Y), Zero_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), Add_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), Mult_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), CBdFuncsNorm(X, Y)$\rangle$.

The following propositions are true:

(19) Let X be a non empty set and Y be a complex normed space. Then $X \longmapsto 0_Y = 0_{\text{the complex normed space of bounded functions from } X \text{ into } Y}$.

(20) Let X be a non empty set, Y be a complex normed space, f be a point of the complex normed space of bounded functions from X into Y, and g be a bounded function from X into the carrier of Y. If $g = f$, then for every element t of X holds $\|g(t)\| \leqslant \|f\|$.

(21) Let X be a non empty set, Y be a complex normed space, and f be a point of the complex normed space of bounded functions from X into Y. Then $0 \leqslant \|f\|$.

(22) Let X be a non empty set, Y be a complex normed space, and f be a point of the complex normed space of bounded functions from X into Y. Suppose $f = 0_{\text{the complex normed space of bounded functions from } X \text{ into } Y}$. Then $0 = \|f\|$.

(23) Let X be a non empty set, Y be a complex normed space, f, g, h be points of the complex normed space of bounded functions from X into Y,

and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(24) Let X be a non empty set, Y be a complex normed space, f, h be points of the complex normed space of bounded functions from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $h' = h$. Let c be a Complex. Then $h = c \cdot f$ if and only if for every element x of X holds $h'(x) = c \cdot f'(x)$.

(25) Let X be a non empty set, Y be a complex normed space, f, g be points of the complex normed space of bounded functions from X into Y, and c be a Complex. Then

(i) $\|f\| = 0$ iff $f = 0_{\text{the complex normed space of bounded functions from } X \text{ into } Y}$,

(ii) $\|c \cdot f\| = |c| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

(26) Let X be a non empty set and Y be a complex normed space. Then the complex normed space of bounded functions from X into Y is complex normed space-like.

(27) Let X be a non empty set and Y be a complex normed space. Then the complex normed space of bounded functions from X into Y is a complex normed space.

Let X be a non empty set and let Y be a complex normed space. One can check that the complex normed space of bounded functions from X into Y is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following three propositions:

(28) Let X be a non empty set, Y be a complex normed space, f, g, h be points of the complex normed space of bounded functions from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f - g$ if and only if for every element x of X holds $h'(x) = f'(x) - g'(x)$.

(29) Let X be a non empty set and Y be a complex normed space. Suppose Y is complete. Let s_1 be a sequence of the complex normed space of bounded functions from X into Y. If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(30) Let X be a non empty set and Y be a complex Banach space. Then the complex normed space of bounded functions from X into Y is a complex Banach space.

Let X be a non empty set and let Y be a complex Banach space. Note that the complex normed space of bounded functions from X into Y is complete.

3. Some Properties of Complex Sequences

We now state four propositions:

(31) For all complex sequences s_2, s_3 such that s_2 is bounded and s_3 is bounded holds $s_2 + s_3$ is bounded.

(32) For every Complex c and for every complex sequence s_1 such that s_1 is bounded holds $c\, s_1$ is bounded.

(33) For every complex sequence s_1 holds s_1 is bounded iff $|s_1|$ is bounded.

(34) For all complex sequences s_2, s_3, s_4 holds $s_2 = s_3 - s_4$ iff for every natural number n holds $s_2(n) = s_3(n) - s_4(n)$.

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Concatenation of Finite Sequences Reducing Overlapping Part and an Argument of Separators of Sequential Files

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Summary. For two finite sequences, we present a notion of their concatenation, reducing overlapping part of the tail of the former and the head of the latter. At the same time, we also give a notion of common part of two finite sequences, which relates to the concatenation given here. A finite sequence is separated by another finite sequence (separator). We examined the condition that a separator separates uniquely any finite sequence. This will become a model of a separator of sequential files.

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The terminology and notation used here are introduced in the following articles: [14], [15], [9], [1], [12], [16], [3], [10], [2], [4], [5], [8], [13], [7], [11], and [6].

The following propositions are true:

(1) For every set D and for every finite sequence f of elements of D holds $f \restriction 0 = \emptyset$.

(2) For every set D and for every finite sequence f of elements of D holds $f_{\downharpoonright 0} = f$.

Let D be a set and let f, g be finite sequences of elements of D. Then $f \frown g$ is a finite sequence of elements of D.

Next we state three propositions:

(3) For every non empty set D and for all finite sequences f, g of elements of D such that $\operatorname{len} f \geqslant 1$ holds $\operatorname{mid}(f \frown g, 1, \operatorname{len} f) = f$.

(4) Let D be a set, f be a finite sequence of elements of D, and i be a natural number. If $i \geqslant \operatorname{len} f$, then $f_{\downharpoonright i} = \varepsilon_D$.

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(5) For every non empty set D and for all natural numbers k_1, k_2 holds $\mathrm{mid}(\varepsilon_D, k_1, k_2) = \varepsilon_D$.

Let D be a set, let f be a finite sequence of elements of D, and let k_1, k_2 be natural numbers. The functor $\mathrm{smid}(f, k_1, k_2)$ yields a finite sequence of elements of D and is defined as follows:

(Def. 1) $\mathrm{smid}(f, k_1, k_2) = f_{\lfloor k_1 -' 1} \restriction ((k_2 + 1) -' k_1)$.

One can prove the following propositions:

(6) Let D be a non empty set, f be a finite sequence of elements of D, and k_1, k_2 be natural numbers. If $k_1 \leqslant k_2$, then $\mathrm{smid}(f, k_1, k_2) = \mathrm{mid}(f, k_1, k_2)$.

(7) Let D be a non empty set, f be a finite sequence of elements of D, and k_2 be a natural number. Then $\mathrm{smid}(f, 1, k_2) = f \restriction k_2$.

(8) Let D be a non empty set, f be a finite sequence of elements of D, and k_2 be a natural number. If $\mathrm{len}\, f \leqslant k_2$, then $\mathrm{smid}(f, 1, k_2) = f$.

(9) Let D be a set, f be a finite sequence of elements of D, and k_1, k_2 be natural numbers. If $k_1 > k_2$, then $\mathrm{smid}(f, k_1, k_2) = \emptyset$ and $\mathrm{smid}(f, k_1, k_2) = \varepsilon_D$.

(10) For every set D and for every finite sequence f of elements of D and for every natural number k_2 holds $\mathrm{smid}(f, 0, k_2) = \mathrm{smid}(f, 1, k_2 + 1)$.

(11) For every non empty set D and for all finite sequences f, g of elements of D holds $\mathrm{smid}(f \frown g, \mathrm{len}\, f + 1, \mathrm{len}\, f + \mathrm{len}\, g) = g$.

Let D be a non empty set and let f, g be finite sequences of elements of D. The functor $\mathrm{ovlpart}(f, g)$ yielding a finite sequence of elements of D is defined by the conditions (Def. 2).

(Def. 2)(i) $\mathrm{len}\,\mathrm{ovlpart}(f, g) \leqslant \mathrm{len}\, g$,

(ii) $\mathrm{ovlpart}(f, g) = \mathrm{smid}(g, 1, \mathrm{len}\,\mathrm{ovlpart}(f, g))$,

(iii) $\mathrm{ovlpart}(f, g) = \mathrm{smid}(f, (\mathrm{len}\, f -' \mathrm{len}\,\mathrm{ovlpart}(f, g)) + 1, \mathrm{len}\, f)$, and

(iv) for every natural number j such that $j \leqslant \mathrm{len}\, g$ and $\mathrm{smid}(g, 1, j) = \mathrm{smid}(f, (\mathrm{len}\, f -' j) + 1, \mathrm{len}\, f)$ holds $j \leqslant \mathrm{len}\,\mathrm{ovlpart}(f, g)$.

Next we state the proposition

(12) For every non empty set D and for all finite sequences f, g of elements of D holds $\mathrm{len}\,\mathrm{ovlpart}(f, g) \leqslant \mathrm{len}\, f$.

Let D be a non empty set and let f, g be finite sequences of elements of D. The functor $\mathrm{ovlcon}(f, g)$ yielding a finite sequence of elements of D is defined as follows:

(Def. 3) $\mathrm{ovlcon}(f, g) = f \frown (g_{\lfloor \mathrm{len}\,\mathrm{ovlpart}(f,g)})$.

One can prove the following proposition

(13) For every non empty set D and for all finite sequences f, g of elements of D holds $\mathrm{ovlcon}(f, g) = (f \restriction (\mathrm{len}\, f -' \mathrm{len}\,\mathrm{ovlpart}(f, g))) \frown g$.

Let D be a non empty set and let f, g be finite sequences of elements of D. The functor ovlldiff(f,g) yields a finite sequence of elements of D and is defined as follows:

(Def. 4) $\operatorname{ovlldiff}(f,g) = f{\upharpoonright}(\operatorname{len} f -' \operatorname{len}\operatorname{ovlpart}(f,g))$.

Let D be a non empty set and let f, g be finite sequences of elements of D. The functor ovlrdiff(f,g) yields a finite sequence of elements of D and is defined by:

(Def. 5) $\operatorname{ovlrdiff}(f,g) = g_{\downharpoonright \operatorname{len}\operatorname{ovlpart}(f,g)}$.

One can prove the following propositions:

(14) Let D be a non empty set and f, g be finite sequences of elements of D. Then $\operatorname{ovlcon}(f,g) = (\operatorname{ovlldiff}(f,g)) \frown \operatorname{ovlpart}(f,g) \frown \operatorname{ovlrdiff}(f,g)$ and $\operatorname{ovlcon}(f,g) = (\operatorname{ovlldiff}(f,g)) \frown ((\operatorname{ovlpart}(f,g)) \frown \operatorname{ovlrdiff}(f,g))$.

(15) Let D be a non empty set and f be a finite sequence of elements of D. Then $\operatorname{ovlcon}(f,f) = f$ and $\operatorname{ovlpart}(f,f) = f$ and $\operatorname{ovlldiff}(f,f) = \emptyset$ and $\operatorname{ovlrdiff}(f,f) = \emptyset$.

(16) For every non empty set D and for all finite sequences f, g of elements of D holds $\operatorname{ovlpart}(f \frown g, g) = g$ and $\operatorname{ovlpart}(f, f \frown g) = f$.

(17) Let D be a non empty set and f, g be finite sequences of elements of D. Then $\operatorname{len}\operatorname{ovlcon}(f,g) = (\operatorname{len} f + \operatorname{len} g) - \operatorname{len}\operatorname{ovlpart}(f,g)$ and $\operatorname{len}\operatorname{ovlcon}(f,g) = (\operatorname{len} f + \operatorname{len} g) -' \operatorname{len}\operatorname{ovlpart}(f,g)$ and $\operatorname{len}\operatorname{ovlcon}(f,g) = \operatorname{len} f + (\operatorname{len} g -' \operatorname{len}\operatorname{ovlpart}(f,g))$.

(18) For every non empty set D and for all finite sequences f, g of elements of D holds $\operatorname{len}\operatorname{ovlpart}(f,g) \leqslant \operatorname{len} f$ and $\operatorname{len}\operatorname{ovlpart}(f,g) \leqslant \operatorname{len} g$.

Let D be a non empty set and let C_1 be a finite sequence of elements of D. We say that C_1 separates uniquely if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let f be a finite sequence of elements of D and i, j be natural numbers. Suppose $1 \leqslant i$ and $i < j$ and $(j + \operatorname{len} C_1) -' 1 \leqslant \operatorname{len} f$ and $\operatorname{smid}(f,i,(i+\operatorname{len} C_1) -' 1) = \operatorname{smid}(f,j,(j+\operatorname{len} C_1) -' 1)$ and $\operatorname{smid}(f,i,(i+\operatorname{len} C_1) -' 1) = C_1$. Then $j -' i \geqslant \operatorname{len} C_1$.

The following proposition is true

(19) Let D be a non empty set and C_1 be a finite sequence of elements of D. Then C_1 separates uniquely if and only if $\operatorname{len}\operatorname{ovlpart}((C_1)_{\downharpoonright 1}, C_1) = 0$.

Let D be a non empty set, let f, g be finite sequences of elements of D, and let n be a natural number. We say that g is a substring of f if and only if:

(Def. 7) If $\operatorname{len} g > 0$, then there exists a natural number i such that $n \leqslant i$ and $i \leqslant \operatorname{len} f$ and $\operatorname{mid}(f,i,(i -' 1) + \operatorname{len} g) = g$.

We now state four propositions:

(20) Let D be a non empty set, f, g be finite sequences of elements of D, and n be a natural number. If $\operatorname{len} g = 0$, then g is a substring of f.

(21) Let D be a non empty set, f, g be finite sequences of elements of D, and n, m be natural numbers. If $m \geqslant n$ and g is a substring of f, then g is a substring of f.

(22) For every non empty set D and for every finite sequence f of elements of D such that $1 \leqslant \operatorname{len} f$ holds f is a substring of f.

(23) Let D be a non empty set and f, g be finite sequences of elements of D. If g is a substring of f, then g is a substring of f.

Let D be a non empty set and let f, g be finite sequences of elements of D. We say that g is a preposition of f if and only if:

(Def. 8) If $\operatorname{len} g > 0$, then $1 \leqslant \operatorname{len} f$ and $\operatorname{mid}(f, 1, \operatorname{len} g) = g$.

One can prove the following four propositions:

(24) Let D be a non empty set and f, g be finite sequences of elements of D. If $\operatorname{len} g = 0$, then g is a preposition of f.

(25) For every non empty set D holds every finite sequence f of elements of D is a preposition of f.

(26) Let D be a non empty set and f, g be finite sequences of elements of D. If g is a preposition of f, then $\operatorname{len} g \leqslant \operatorname{len} f$.

(27) Let D be a non empty set and f, g be finite sequences of elements of D. If $\operatorname{len} g > 0$ and g is a preposition of f, then $g(1) = f(1)$.

Let D be a non empty set and let f, g be finite sequences of elements of D. We say that g is a postposition of f if and only if:

(Def. 9) $\operatorname{Rev}(g)$ is a preposition of $\operatorname{Rev}(f)$.

Next we state several propositions:

(28) Let D be a non empty set and f, g be finite sequences of elements of D. If $\operatorname{len} g = 0$, then g is a postposition of f.

(29) Let D be a non empty set and f, g be finite sequences of elements of D. If g is a postposition of f, then $\operatorname{len} g \leqslant \operatorname{len} f$.

(30) Let D be a non empty set, f, g be finite sequences of elements of D, and n be a natural number. Suppose g is a postposition of f. If $\operatorname{len} g > 0$, then $\operatorname{len} g \leqslant \operatorname{len} f$ and $\operatorname{mid}(f, (\operatorname{len} f + 1) -' \operatorname{len} g, \operatorname{len} f) = g$.

(31) Let D be a non empty set, f, g be finite sequences of elements of D, and n be a natural number such that if $\operatorname{len} g > 0$, then $\operatorname{len} g \leqslant \operatorname{len} f$ and $\operatorname{mid}(f, (\operatorname{len} f + 1) -' \operatorname{len} g, \operatorname{len} f) = g$. Then g is a postposition of f.

(32) Let D be a non empty set, f, g be finite sequences of elements of D, and n be a natural number. If $\operatorname{len} g = 0$, then g is a preposition of f.

(33) Let D be a non empty set, f, g be finite sequences of elements of D, and n be a natural number. If $1 \leqslant \operatorname{len} f$ and g is a preposition of f, then g is

a substring of f.

(34) Let D be a non empty set, f, g be finite sequences of elements of D, and n be a natural number. Suppose g is not a substring of f. Let i be a natural number. If $n \leqslant i$ and $0 < i$, then $\mathrm{mid}(f, i, (i -' 1) + \mathrm{len}\, g) \neq g$.

Let D be a non empty set, let f, g be finite sequences of elements of D, and let n be a natural number. The functor $\mathrm{instr}(n, f)$ yielding a natural number is defined by the conditions (Def. 10).

(Def. 10)(i) If $\mathrm{instr}(n, f) \neq 0$, then $n \leqslant \mathrm{instr}(n, f)$ and g is a preposition of $f_{\lfloor \mathrm{instr}(n,f) -' 1}$ and for every natural number j such that $j \geqslant n$ and $j > 0$ and g is a preposition of $f_{\lfloor j -' 1}$ holds $j \geqslant \mathrm{instr}(n, f)$, and

(ii) if $\mathrm{instr}(n, f) = 0$, then g is not a substring of f.

Let D be a non empty set and let f, C_1 be finite sequences of elements of D. The functor $\mathrm{addcr}(f, C_1)$ yields a finite sequence of elements of D and is defined by:

(Def. 11) $\mathrm{addcr}(f, C_1) = \mathrm{ovlcon}(f, C_1)$.

Let D be a non empty set and let r, C_1 be finite sequences of elements of D. We say that r is terminated by C_1 if and only if:

(Def. 12) If $\mathrm{len}\, C_1 > 0$, then $\mathrm{len}\, r \geqslant \mathrm{len}\, C_1$ and $\mathrm{instr}(1, r) = (\mathrm{len}\, r + 1) -' \mathrm{len}\, C_1$.

The following proposition is true

(35) For every non empty set D holds every finite sequence f of elements of D is terminated by f.

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Cauchy Sequence of Complex Unitary Space

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Summary. As an extension of [13], we introduce the Cauchy sequence of complex unitary space and describe its properties.

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The terminology and notation used in this paper are introduced in the following papers: [22], [3], [20], [9], [5], [12], [10], [11], [15], [2], [18], [4], [1], [21], [16], [17], [14], [13], [19], [6], [7], and [8].

For simplicity, we follow the rules: X denotes a complex unitary space, s_1, s_2, s_3 denote sequences of X, R_1 denotes a sequence of real numbers, C_1, C_2, C_3 denote complex sequences, z, z_1, z_2 denote Complexes, r denotes a real number, and k, n, m denote natural numbers.

The scheme *Rec Func Ex CUS* deals with a complex unitary space $\mathcal{A}$, a point $\mathcal{B}$ of $\mathcal{A}$, and a binary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

> There exists a function f from $\mathbb{N}$ into the carrier of $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every element n of $\mathbb{N}$ and for every point x of $\mathcal{A}$ such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$

for all values of the parameters.

Let us consider X, s_1. The functor $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ yields a sequence of X and is defined as follows:

(Def. 1) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(0) = s_1(0)$ and for every n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + s_1(n+1)$.

One can prove the following propositions:

(1) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2+s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(2) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2-s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(3) $(\sum_{\alpha=0}^{\kappa}(z\cdot s_1)(\alpha))_{\kappa\in\mathbb{N}} = z\cdot(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

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(4) $(\sum_{\alpha=0}^{\kappa}(-s_1)(\alpha))_{\kappa\in\mathbb{N}} = -(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(5) $z_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} + z_2 \cdot (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(z_1 \cdot s_2 + z_2 \cdot s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

Let us consider X, s_1. We say that s_1 is summable if and only if:

(Def. 2) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

The functor $\sum s_1$ yields a point of X and is defined as follows:

(Def. 3) $\sum s_1 = \lim((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$.

Next we state several propositions:

(6) If s_2 is summable and s_3 is summable, then $s_2 + s_3$ is summable and $\sum(s_2 + s_3) = \sum s_2 + \sum s_3$.

(7) If s_2 is summable and s_3 is summable, then $s_2 - s_3$ is summable and $\sum(s_2 - s_3) = \sum s_2 - \sum s_3$.

(8) If s_1 is summable, then $z \cdot s_1$ is summable and $\sum(z \cdot s_1) = z \cdot \sum s_1$.

(9) If s_1 is summable, then s_1 is convergent and $\lim s_1 = 0_X$.

(10) Suppose X is Hilbert. Then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m)\| < r$.

(11) If s_1 is summable, then $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is bounded.

(12) If for every n holds $s_2(n) = s_1(0)$, then $(\sum_{\alpha=0}^{\kappa}(s_1 \uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}} \uparrow 1 - s_2$.

(13) If s_1 is summable, then for every k holds $s_1 \uparrow k$ is summable.

(14) If there exists k such that $s_1 \uparrow k$ is summable, then s_1 is summable.

Let us consider X, s_1, n. The functor $\sum_{\kappa=0}^{n} s_1(\kappa)$ yielding a point of X is defined by:

(Def. 4) $\sum_{\kappa=0}^{n} s_1(\kappa) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

One can prove the following propositions:

(15) $\sum_{\kappa=0}^{0} s_1(\kappa) = s_1(0)$.

(16) $\sum_{\kappa=0}^{1} s_1(\kappa) = \sum_{\kappa=0}^{0} s_1(\kappa) + s_1(1)$.

(17) $\sum_{\kappa=0}^{1} s_1(\kappa) = s_1(0) + s_1(1)$.

(18) $\sum_{\kappa=0}^{n+1} s_1(\kappa) = \sum_{\kappa=0}^{n} s_1(\kappa) + s_1(n+1)$.

(19) $s_1(n+1) = \sum_{\kappa=0}^{n+1} s_1(\kappa) - \sum_{\kappa=0}^{n} s_1(\kappa)$.

(20) $s_1(1) = \sum_{\kappa=0}^{1} s_1(\kappa) - \sum_{\kappa=0}^{0} s_1(\kappa)$.

Let us consider X, s_1, n, m. The functor $\sum_{\kappa=n+1}^{m} s_1(\kappa)$ yielding a point of X is defined by:

(Def. 5) $\sum_{\kappa=n+1}^{m} s_1(\kappa) = \sum_{\kappa=0}^{n} s_1(\kappa) - \sum_{\kappa=0}^{m} s_1(\kappa)$.

One can prove the following four propositions:

(21) $\sum_{\kappa=1+1}^{0} s_1(\kappa) = s_1(1)$.

(22) $\sum_{\kappa=n+1+1}^{n} s_1(\kappa) = s_1(n+1)$.

(23) Suppose X is Hilbert. Then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|\sum_{\kappa=0}^{n} s_1(\kappa) - \sum_{\kappa=0}^{m} s_1(\kappa)\| < r$.

(24) Suppose X is Hilbert. Then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|\sum_{\kappa=n+1}^{m} s_1(\kappa)\| < r$.

Let us consider C_1, n. The functor $\sum_{\kappa=0}^{n} C_1(\kappa)$ yielding a Complex is defined as follows:

(Def. 6) $\sum_{\kappa=0}^{n} C_1(\kappa) = (\sum_{\alpha=0}^{\kappa}(C_1)(\alpha))_{\kappa \in \mathbb{N}}(n)$.

Let us consider C_1, n, m. The functor $\sum_{\kappa=n+1}^{m} C_1(\kappa)$ yielding a Complex is defined by:

(Def. 7) $\sum_{\kappa=n+1}^{m} C_1(\kappa) = \sum_{\kappa=0}^{n} C_1(\kappa) - \sum_{\kappa=0}^{m} C_1(\kappa)$.

Let us consider X, s_1. We say that s_1 is absolutely summable if and only if:

(Def. 8) $\|s_1\|$ is summable.

The following propositions are true:

(25) If s_2 is absolutely summable and s_3 is absolutely summable, then $s_2 + s_3$ is absolutely summable.

(26) If s_1 is absolutely summable, then $z \cdot s_1$ is absolutely summable.

(27) If for every n holds $\|s_1\|(n) \leqslant R_1(n)$ and R_1 is summable, then s_1 is absolutely summable.

(28) If for every n holds $s_1(n) \neq 0_X$ and $R_1(n) = \frac{\|s_1(n+1)\|}{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 < 1$, then s_1 is absolutely summable.

(29) If $r > 0$ and there exists m such that for every n such that $n \geqslant m$ holds $\|s_1(n)\| \geqslant r$, then s_1 is not convergent or $\lim s_1 \neq 0_X$.

(30) If for every n holds $s_1(n) \neq 0_X$ and there exists m such that for every n such that $n \geqslant m$ holds $\frac{\|s_1(n+1)\|}{\|s_1(n)\|} \geqslant 1$, then s_1 is not summable.

(31) If for every n holds $s_1(n) \neq 0_X$ and for every n holds $R_1(n) = \frac{\|s_1(n+1)\|}{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 > 1$, then s_1 is not summable.

(32) If for every n holds $R_1(n) = \sqrt[n]{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 < 1$, then s_1 is absolutely summable.

(33) If for every n holds $R_1(n) = \sqrt[n]{\|s_1\|(n)}$ and there exists m such that for every n such that $n \geqslant m$ holds $R_1(n) \geqslant 1$, then s_1 is not summable.

(34) If for every n holds $R_1(n) = \sqrt[n]{\|s_1\|(n)}$ and R_1 is convergent and $\lim R_1 > 1$, then s_1 is not summable.

(35) $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa \in \mathbb{N}}$ is non-decreasing.

(36) For every n holds $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa \in \mathbb{N}}(n) \geqslant 0$.

(37) For every n holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(n)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(38) For every n holds $\|\sum_{\kappa=0}^{n} s_1(\kappa)\| \leqslant \sum_{\kappa=0}^{n}\|s_1\|(\kappa)$.

(39) For all n, m holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leqslant |(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n)|$.

(40) For all n, m holds $\|\sum_{\kappa=0}^{m} s_1(\kappa) - \sum_{\kappa=0}^{n} s_1(\kappa)\| \leqslant |\sum_{\kappa=0}^{m}\|s_1\|(\kappa) - \sum_{\kappa=0}^{n}\|s_1\|(\kappa)|$.

(41) For all n, m holds $\|\sum_{\kappa=m+1}^{n} s_1(\kappa)\| \leqslant |\sum_{\kappa=m+1}^{n}\|s_1\|(\kappa)|$.

(42) If X is Hilbert, then if s_1 is absolutely summable, then s_1 is summable.

Let us consider X, s_1, C_1. The functor $C_1 \cdot s_1$ yields a sequence of X and is defined by:

(Def. 9) For every n holds $(C_1 \cdot s_1)(n) = C_1(n) \cdot s_1(n)$.

Next we state several propositions:

(43) $C_1 \cdot (s_2 + s_3) = C_1 \cdot s_2 + C_1 \cdot s_3$.

(44) $(C_2 + C_3) \cdot s_1 = C_2 \cdot s_1 + C_3 \cdot s_1$.

(45) $(C_2\, C_3) \cdot s_1 = C_2 \cdot (C_3 \cdot s_1)$.

(46) $(z\, C_1) \cdot s_1 = z \cdot (C_1 \cdot s_1)$.

(47) $C_1 \cdot -s_1 = (-C_1) \cdot s_1$.

(48) If C_1 is convergent and s_1 is convergent, then $C_1 \cdot s_1$ is convergent.

(49) If C_1 is bounded and s_1 is bounded, then $C_1 \cdot s_1$ is bounded.

(50) If C_1 is convergent and s_1 is convergent, then $C_1 \cdot s_1$ is convergent and $\lim(C_1 \cdot s_1) = \lim C_1 \cdot \lim s_1$.

Let us consider C_1. We say that C_1 is Cauchy if and only if:

(Def. 10) For every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $|C_1(n) - C_1(m)| < r$.

We introduce C_1 is a Cauchy sequence as a synonym of C_1 is Cauchy.

Next we state four propositions:

(51) If X is Hilbert, then if s_1 is Cauchy and C_1 is Cauchy, then $C_1 \cdot s_1$ is Cauchy.

(52) For every n holds $(\sum_{\alpha=0}^{\kappa}((C_1 - C_1 \uparrow 1) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa}(C_1 \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) - (C_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(n+1)$.

(53) For every n holds $(\sum_{\alpha=0}^{\kappa}(C_1 \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (C_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(n+1) - (\sum_{\alpha=0}^{\kappa}((C_1 \uparrow 1 - C_1) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(54) For every n holds $\sum_{\kappa=0}^{n+1}(C_1 \cdot s_1)(\kappa) = (C_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(n+1) - \sum_{\kappa=0}^{n}((C_1 \uparrow 1 - C_1) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(\kappa)$.

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