

# Banach Space of Bounded Linear Operators

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**Summary.** In this article, the basic properties of linear spaces which are defined as the set of all linear operators from one linear space to another, are described. Especially, the Banach space is introduced. This is defined by the set of all bounded linear operators.

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The notation and terminology used in this paper are introduced in the following articles: [26], [6], [24], [31], [27], [33], [32], [4], [5], [16], [23], [22], [3], [1], [2], [21], [28], [9], [7], [30], [14], [25], [17], [29], [19], [18], [8], [20], [13], [11], [12], [10], and [15].

## 1. REAL VECTOR SPACE OF OPERATORS

Let  $X$  be a set, let  $Y$  be a non empty set, let  $F$  be a function from  $[\mathbb{R}, Y]$  into  $Y$ , let  $a$  be a real number, and let  $f$  be a function from  $X$  into  $Y$ . Then  $F^\circ(a, f)$  is an element of  $Y^X$ .

One can prove the following propositions:

- (1) Let  $X$  be a non empty set and  $Y$  be a non empty loop structure. Then there exists a binary operation  $A_1$  on  $(\text{the carrier of } Y)^X$  such that for all elements  $f, g$  of  $(\text{the carrier of } Y)^X$  holds  $A_1(f, g) = (\text{the addition of } Y)^\circ(f, g)$ .
- (2) Let  $X$  be a non empty set and  $Y$  be a real linear space. Then there exists a function  $M_1$  from  $[\mathbb{R}, (\text{the carrier of } Y)^X]$  into  $(\text{the carrier of } Y)^X$  such that for every real number  $r$  and for every element  $f$  of  $(\text{the carrier of } Y)^X$  and for every element  $s$  of  $X$  holds  $M_1(\langle r, f \rangle)(s) = r \cdot f(s)$ .

Let  $X$  be a non empty set and let  $Y$  be a non empty loop structure. The functor  $\text{FuncAdd}(X, Y)$  yields a binary operation on  $(\text{the carrier of } Y)^X$  and is defined by:

- (Def. 1) For all elements  $f, g$  of  $(\text{the carrier of } Y)^X$  holds  $(\text{FuncAdd}(X, Y))(f, g) = (\text{the addition of } Y)^\circ(f, g)$ .

Let  $X$  be a non empty set and let  $Y$  be a real linear space. The functor  $\text{FuncExtMult}(X, Y)$  yields a function from  $[\mathbb{R}, (\text{the carrier of } Y)^X]$  into  $(\text{the carrier of } Y)^X$  and is defined by the condition (Def. 2).

- (Def. 2) Let  $a$  be a real number,  $f$  be an element of  $(\text{the carrier of } Y)^X$ , and  $x$  be an element of  $X$ . Then  $(\text{FuncExtMult}(X, Y))(\langle a, f \rangle)(x) = a \cdot f(x)$ .

Let  $X$  be a set and let  $Y$  be a non empty zero structure. The functor  $\text{FuncZero}(X, Y)$  yielding an element of  $(\text{the carrier of } Y)^X$  is defined as follows:

- (Def. 3)  $\text{FuncZero}(X, Y) = X \mapsto 0_Y$ .

We adopt the following rules:  $X$  is a non empty set,  $Y$  is a real linear space, and  $f, g, h$  are elements of  $(\text{the carrier of } Y)^X$ .

The following two propositions are true:

- (3) Let  $Y$  be a non empty loop structure and  $f, g, h$  be elements of  $(\text{the carrier of } Y)^X$ . Then  $h = (\text{FuncAdd}(X, Y))(f, g)$  if and only if for every element  $x$  of  $X$  holds  $h(x) = f(x) + g(x)$ .
- (4) For every element  $x$  of  $X$  holds  $(\text{FuncZero}(X, Y))(x) = 0_Y$ .

In the sequel  $a, b$  are real numbers.

The following propositions are true:

- (5)  $h = (\text{FuncExtMult}(X, Y))(\langle a, f \rangle)$  iff for every element  $x$  of  $X$  holds  $h(x) = a \cdot f(x)$ .
- (6)  $(\text{FuncAdd}(X, Y))(f, g) = (\text{FuncAdd}(X, Y))(g, f)$ .
- (7)  $(\text{FuncAdd}(X, Y))(f, (\text{FuncAdd}(X, Y))(g, h)) = (\text{FuncAdd}(X, Y))((\text{FuncAdd}(X, Y))(f, g), h)$ .
- (8)  $(\text{FuncAdd}(X, Y))(\text{FuncZero}(X, Y), f) = f$ .
- (9)  $(\text{FuncAdd}(X, Y))(f, (\text{FuncExtMult}(X, Y))(\langle -1, f \rangle)) = \text{FuncZero}(X, Y)$ .
- (10)  $(\text{FuncExtMult}(X, Y))(\langle 1, f \rangle) = f$ .
- (11)  $(\text{FuncExtMult}(X, Y))(\langle a, (\text{FuncExtMult}(X, Y))(\langle b, f \rangle) \rangle) = (\text{FuncExtMult}(X, Y))(\langle a \cdot b, f \rangle)$ .
- (12)  $(\text{FuncAdd}(X, Y))((\text{FuncExtMult}(X, Y))(\langle a, f \rangle), (\text{FuncExtMult}(X, Y))(\langle b, f \rangle)) = (\text{FuncExtMult}(X, Y))(\langle a + b, f \rangle)$ .
- (13)  $\langle (\text{the carrier of } Y)^X, \text{FuncZero}(X, Y), \text{FuncAdd}(X, Y), \text{FuncExtMult}(X, Y) \rangle$  is a real linear space.

Let  $X$  be a non empty set and let  $Y$  be a real linear space. The functor  $\text{RealVectSpace}(X, Y)$  yields a real linear space and is defined as follows:

(Def. 4)  $\text{RealVectSpace}(X, Y) = \langle (\text{the carrier of } Y)^X, \text{FuncZero}(X, Y), \text{FuncAdd}(X, Y), \text{FuncExtMult}(X, Y) \rangle$ .

Let  $X$  be a non empty set and let  $Y$  be a real linear space. One can check that  $\text{RealVectSpace}(X, Y)$  is strict.

Let  $X$  be a non empty set and let  $Y$  be a real linear space. Note that every vector of  $\text{RealVectSpace}(X, Y)$  is function-like and relation-like.

Let  $X$  be a non empty set, let  $Y$  be a real linear space, let  $f$  be a vector of  $\text{RealVectSpace}(X, Y)$ , and let  $x$  be an element of  $X$ . Then  $f(x)$  is a vector of  $Y$ .

One can prove the following propositions:

- (14) Let  $X$  be a non empty set,  $Y$  be a real linear space, and  $f, g, h$  be vectors of  $\text{RealVectSpace}(X, Y)$ . Then  $h = f + g$  if and only if for every element  $x$  of  $X$  holds  $h(x) = f(x) + g(x)$ .
- (15) Let  $X$  be a non empty set,  $Y$  be a real linear space,  $f, h$  be vectors of  $\text{RealVectSpace}(X, Y)$ , and  $a$  be a real number. Then  $h = a \cdot f$  if and only if for every element  $x$  of  $X$  holds  $h(x) = a \cdot f(x)$ .
- (16) For every non empty set  $X$  and for every real linear space  $Y$  holds  $0_{\text{RealVectSpace}(X, Y)} = X \mapsto 0_Y$ .

## 2. REAL VECTOR SPACE OF LINEAR OPERATORS

Let  $X$  be a non empty RLS structure, let  $Y$  be a non empty loop structure, and let  $I_1$  be a function from  $X$  into  $Y$ . We say that  $I_1$  is additive if and only if:

(Def. 5) For all vectors  $x, y$  of  $X$  holds  $I_1(x + y) = I_1(x) + I_1(y)$ .

Let  $X, Y$  be non empty RLS structures and let  $I_1$  be a function from  $X$  into  $Y$ . We say that  $I_1$  is homogeneous if and only if:

(Def. 6) For every vector  $x$  of  $X$  and for every real number  $r$  holds  $I_1(r \cdot x) = r \cdot I_1(x)$ .

Let  $X$  be a non empty RLS structure and let  $Y$  be a real linear space. Note that there exists a function from  $X$  into  $Y$  which is additive and homogeneous.

Let  $X, Y$  be real linear spaces. A linear operator from  $X$  into  $Y$  is an additive homogeneous function from  $X$  into  $Y$ .

Let  $X, Y$  be real linear spaces. The functor  $\text{LinearOperators}(X, Y)$  yields a subset of  $\text{RealVectSpace}(\text{the carrier of } X, Y)$  and is defined as follows:

(Def. 7) For every set  $x$  holds  $x \in \text{LinearOperators}(X, Y)$  iff  $x$  is a linear operator from  $X$  into  $Y$ .

Let  $X, Y$  be real linear spaces. Note that  $\text{LinearOperators}(X, Y)$  is non empty.

One can prove the following propositions:

- (17) For all real linear spaces  $X, Y$  holds  $\text{LinearOperators}(X, Y)$  is linearly closed.
- (18) Let  $X, Y$  be real linear spaces. Then  $\langle \text{LinearOperators}(X, Y), \text{Zero}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)), \text{Add}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)), \text{Mult}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)) \rangle$  is a subspace of  $\text{RealVectSpace}(\text{the carrier of } X, Y)$ .

Let  $X, Y$  be real linear spaces. One can verify that  $\langle \text{LinearOperators}(X, Y), \text{Zero}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)), \text{Add}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)), \text{Mult}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)) \rangle$  is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

One can prove the following proposition

- (19) Let  $X, Y$  be real linear spaces. Then  $\langle \text{LinearOperators}(X, Y), \text{Zero}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)), \text{Add}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)), \text{Mult}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)) \rangle$  is a real linear space.

Let  $X, Y$  be real linear spaces. The functor  $\text{RVectorSpaceOfLinearOperators}(X, Y)$  yielding a real linear space is defined as follows:

- (Def. 8)  $\text{RVectorSpaceOfLinearOperators}(X, Y) = \langle \text{LinearOperators}(X, Y), \text{Zero}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)), \text{Add}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)), \text{Mult}_{\cdot}(\text{LinearOperators}(X, Y), \text{RealVectSpace}(\text{the carrier of } X, Y)) \rangle$ .

Let  $X, Y$  be real linear spaces. Observe that  $\text{RVectorSpaceOfLinearOperators}(X, Y)$  is strict.

Let  $X, Y$  be real linear spaces. Note that every element of  $\text{RVectorSpaceOfLinearOperators}(X, Y)$  is function-like and relation-like.

Let  $X, Y$  be real linear spaces, let  $f$  be an element of

$\text{RVectorSpaceOfLinearOperators}(X, Y)$ , and let  $v$  be a vector of  $X$ . Then  $f(v)$  is a vector of  $Y$ .

We now state four propositions:

- (20) Let  $X, Y$  be real linear spaces and  $f, g, h$  be vectors of  $\text{RVectorSpaceOfLinearOperators}(X, Y)$ . Then  $h = f + g$  if and only if for every vector  $x$  of  $X$  holds  $h(x) = f(x) + g(x)$ .
- (21) Let  $X, Y$  be real linear spaces,  $f, h$  be vectors of  $\text{RVectorSpaceOfLinearOperators}(X, Y)$ , and  $a$  be a real number. Then  $h = a \cdot f$  if and only if for every vector  $x$  of  $X$  holds  $h(x) = a \cdot f(x)$ .

- (22) For all real linear spaces  $X, Y$  holds  $0_{\text{RVectorSpaceOfLinearOperators}(X,Y)} =$   
(the carrier of  $X$ )  $\mapsto 0_Y$ .
- (23) For all real linear spaces  $X, Y$  holds (the carrier of  $X$ )  $\mapsto 0_Y$  is a linear  
operator from  $X$  into  $Y$ .

### 3. REAL NORMED LINEAR SPACE OF BOUNDED LINEAR OPERATORS

One can prove the following proposition

- (24) Let  $X$  be a real normed space,  $s_1$  be a sequence of  $X$ , and  $g$  be a point  
of  $X$ . If  $s_1$  is convergent and  $\lim s_1 = g$ , then  $\|s_1\|$  is convergent and  
 $\lim \|s_1\| = \|g\|$ .

Let  $X, Y$  be real normed spaces and let  $I_1$  be a linear operator from  $X$  into  
 $Y$ . We say that  $I_1$  is bounded if and only if:

- (Def. 9) There exists a real number  $K$  such that  $0 \leq K$  and for every vector  $x$  of  
 $X$  holds  $\|I_1(x)\| \leq K \cdot \|x\|$ .

Next we state the proposition

- (25) Let  $X, Y$  be real normed spaces and  $f$  be a linear operator from  $X$  into  
 $Y$ . If for every vector  $x$  of  $X$  holds  $f(x) = 0_Y$ , then  $f$  is bounded.

Let  $X, Y$  be real normed spaces. One can check that there exists a linear  
operator from  $X$  into  $Y$  which is bounded.

Let  $X, Y$  be real normed spaces. The functor  $\text{BoundedLinearOperators}(X, Y)$   
yields a subset of  $\text{RVectorSpaceOfLinearOperators}(X, Y)$  and is defined by:

- (Def. 10) For every set  $x$  holds  $x \in \text{BoundedLinearOperators}(X, Y)$  iff  $x$  is a bo-  
unded linear operator from  $X$  into  $Y$ .

Let  $X, Y$  be real normed spaces. One can verify that  $\text{BoundedLinearOperators}$   
 $(X, Y)$  is non empty.

One can prove the following two propositions:

- (26) For all real normed spaces  $X, Y$  holds  $\text{BoundedLinearOperators}(X, Y)$   
is linearly closed.
- (27) For all real normed spaces  $X, Y$  holds  $\langle \text{BoundedLinearOperators}(X, Y),$   
 $\text{Zero}_-(\text{BoundedLinearOperators}(X, Y), \text{RVectorSpaceOfLinearOperators}$   
 $(X, Y)), \text{Add}_-(\text{BoundedLinearOperators}(X, Y),$   
 $\text{RVectorSpaceOfLinearOperators}(X, Y)), \text{Mult}_-(\text{BoundedLinearOperators}$   
 $(X, Y), \text{RVectorSpaceOfLinearOperators}(X, Y)) \rangle$  is a subspace  
of  $\text{RVectorSpaceOfLinearOperators}(X, Y)$ .

Let  $X, Y$  be real normed spaces.

Observe that  $\langle \text{BoundedLinearOperators}(X, Y),$   
 $\text{Zero}_-(\text{BoundedLinearOperators}(X, Y), \text{RVectorSpaceOfLinearOperators}(X,$   
 $Y)), \text{Add}_-(\text{BoundedLinearOperators}(X, Y), \text{RVectorSpaceOfLinearOperators}$   
 $(X, Y)), \text{Mult}_-(\text{BoundedLinearOperators}(X, Y),$

$\text{RVectorSpaceOfLinearOperators}(X, Y))$  is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

One can prove the following proposition

- (28) For all real normed spaces  $X, Y$  holds  $\langle \text{BoundedLinearOperators}(X, Y), \text{Zero}(\text{BoundedLinearOperators}(X, Y), \text{RVectorSpaceOfLinearOperators}(X, Y)), \text{Add}(\text{BoundedLinearOperators}(X, Y), \text{RVectorSpaceOfLinearOperators}(X, Y)), \text{Mult}(\text{BoundedLinearOperators}(X, Y), \text{RVectorSpaceOfLinearOperators}(X, Y)) \rangle$  is a real linear space.

Let  $X, Y$  be real normed spaces.

The functor  $\text{RVectorSpaceOfBoundedLinearOperators}(X, Y)$  yields a real linear space and is defined by:

- (Def. 11)  $\text{RVectorSpaceOfBoundedLinearOperators}(X, Y) = \langle \text{BoundedLinearOperators}(X, Y), \text{Zero}(\text{BoundedLinearOperators}(X, Y), \text{RVectorSpaceOfLinearOperators}(X, Y)), \text{Add}(\text{BoundedLinearOperators}(X, Y), \text{RVectorSpaceOfLinearOperators}(X, Y)), \text{Mult}(\text{BoundedLinearOperators}(X, Y), \text{RVectorSpaceOfLinearOperators}(X, Y)) \rangle$ .

Let  $X, Y$  be real normed spaces.

Observe that  $\text{RVectorSpaceOfBoundedLinearOperators}(X, Y)$  is strict.

Let  $X, Y$  be real normed spaces. Note that every element of  $\text{RVectorSpaceOfBoundedLinearOperators}(X, Y)$  is function-like and relation-like.

Let  $X, Y$  be real normed spaces, let  $f$  be an element of  $\text{RVectorSpaceOfBoundedLinearOperators}(X, Y)$ , and let  $v$  be a vector of  $X$ . Then  $f(v)$  is a vector of  $Y$ .

One can prove the following propositions:

- (29) Let  $X, Y$  be real normed spaces and  $f, g, h$  be vectors of  $\text{RVectorSpaceOfBoundedLinearOperators}(X, Y)$ . Then  $h = f + g$  if and only if for every vector  $x$  of  $X$  holds  $h(x) = f(x) + g(x)$ .
- (30) Let  $X, Y$  be real normed spaces,  $f, h$  be vectors of  $\text{RVectorSpaceOfBoundedLinearOperators}(X, Y)$ , and  $a$  be a real number. Then  $h = a \cdot f$  if and only if for every vector  $x$  of  $X$  holds  $h(x) = a \cdot f(x)$ .
- (31) For all real normed spaces  $X, Y$  holds

$$0_{\text{RVectorSpaceOfBoundedLinearOperators}(X, Y)} = (\text{the carrier of } X) \mapsto 0_Y.$$

Let  $X, Y$  be real normed spaces and let  $f$  be a set. Let us assume that  $f \in \text{BoundedLinearOperators}(X, Y)$ . The functor  $\text{modetrans}(f, X, Y)$  yields a bounded linear operator from  $X$  into  $Y$  and is defined by:

- (Def. 12)  $\text{modetrans}(f, X, Y) = f$ .

Let  $X, Y$  be real normed spaces and let  $u$  be a linear operator from  $X$  into  $Y$ . The functor  $\text{PreNorms}(u)$  yielding a non empty subset of  $\mathbb{R}$  is defined as follows:

(Def. 13)  $\text{PreNorms}(u) = \{\|u(t)\|; t \text{ ranges over vectors of } X: \|t\| \leq 1\}$ .

We now state three propositions:

(32) Let  $X, Y$  be real normed spaces and  $g$  be a bounded linear operator from  $X$  into  $Y$ . Then  $\text{PreNorms}(g)$  is non empty and upper bounded.

(33) Let  $X, Y$  be real normed spaces and  $g$  be a linear operator from  $X$  into  $Y$ . Then  $g$  is bounded if and only if  $\text{PreNorms}(g)$  is upper bounded.

(34) Let  $X, Y$  be real normed spaces. Then there exists a function  $N_1$  from  $\text{BoundedLinearOperators}(X, Y)$  into  $\mathbb{R}$  such that for every set  $f$  if  $f \in \text{BoundedLinearOperators}(X, Y)$ , then  $N_1(f) = \sup \text{PreNorms}(\text{modetrans}(f, X, Y))$ .

Let  $X, Y$  be real normed spaces. The functor  $\text{BoundedLinearOperatorsNorm}(X, Y)$  yielding a function from  $\text{BoundedLinearOperators}(X, Y)$  into  $\mathbb{R}$  is defined as follows:

(Def. 14) For every set  $x$  such that  $x \in \text{BoundedLinearOperators}(X, Y)$  holds  $(\text{BoundedLinearOperatorsNorm}(X, Y))(x) = \sup \text{PreNorms}(\text{modetrans}(x, X, Y))$ .

The following two propositions are true:

(35) For all real normed spaces  $X, Y$  and for every bounded linear operator  $f$  from  $X$  into  $Y$  holds  $\text{modetrans}(f, X, Y) = f$ .

(36) For all real normed spaces  $X, Y$  and for every bounded linear operator  $f$  from  $X$  into  $Y$  holds  $(\text{BoundedLinearOperatorsNorm}(X, Y))(f) = \sup \text{PreNorms}(f)$ .

Let  $X, Y$  be real normed spaces.

The functor  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$  yielding a non empty normed structure is defined as follows:

(Def. 15)  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y) = \langle \text{BoundedLinearOperators}(X, Y), \text{Zero}_{\text{BoundedLinearOperators}(X, Y)}, \text{RVectorSpaceOfLinearOperators}(X, Y), \text{Add}_{\text{BoundedLinearOperators}(X, Y)}, \text{RVectorSpaceOfLinearOperators}(X, Y), \text{Mult}_{\text{BoundedLinearOperators}(X, Y)}, \text{RVectorSpaceOfLinearOperators}(X, Y), \text{BoundedLinearOperatorsNorm}(X, Y) \rangle$ .

The following propositions are true:

(37) For all real normed spaces  $X, Y$  holds  $(\text{the carrier of } X) \mapsto 0_Y = {}^0_{\text{RNormSpaceOfBoundedLinearOperators}(X, Y)}$ .

(38) Let  $X, Y$  be real normed spaces,  $f$  be a point

of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$ , and  $g$  be a bounded linear operator from  $X$  into  $Y$ . If  $g = f$ , then for every vector  $t$  of  $X$  holds  $\|g(t)\| \leq \|f\| \cdot \|t\|$ .

(39) For all real normed spaces  $X, Y$  and for every point  $f$  of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$  holds  $0 \leq \|f\|$ .

(40) For all real normed spaces  $X, Y$  and for every point  $f$  of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$  such that  $f = 0_{\text{RNormSpaceOfBoundedLinearOperators}(X, Y)}$  holds  $0 = \|f\|$ .

Let  $X, Y$  be real normed spaces. Observe that every element of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$  is function-like and relation-like.

Let  $X, Y$  be real normed spaces, let  $f$  be an element of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$ , and let  $v$  be a vector of  $X$ . Then  $f(v)$  is a vector of  $Y$ .

The following propositions are true:

(41) Let  $X, Y$  be real normed spaces and  $f, g, h$  be points of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$ . Then  $h = f + g$  if and only if for every vector  $x$  of  $X$  holds  $h(x) = f(x) + g(x)$ .

(42) Let  $X, Y$  be real normed spaces,  $f, h$  be points of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$ , and  $a$  be a real number. Then  $h = a \cdot f$  if and only if for every vector  $x$  of  $X$  holds  $h(x) = a \cdot f(x)$ .

(43) Let  $X$  be a real normed space,  $Y$  be a real normed space,  $f, g$  be points of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$ , and  $a$  be a real number. Then  $\|f\| = 0$  iff  $f = 0_{\text{RNormSpaceOfBoundedLinearOperators}(X, Y)}$  and  $\|a \cdot f\| = |a| \cdot \|f\|$  and  $\|f + g\| \leq \|f\| + \|g\|$ .

(44) For all real normed spaces  $X, Y$  holds  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$  is real normed space-like.

(45) For all real normed spaces  $X, Y$  holds  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$  is a real normed space.

Let  $X, Y$  be real normed spaces.

Note that  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$  is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following proposition

(46) Let  $X, Y$  be real normed spaces and  $f, g, h$  be points of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$ . Then  $h = f - g$  if and only if for every vector  $x$  of  $X$  holds  $h(x) = f(x) - g(x)$ .



## 4. REAL BANACH SPACE OF BOUNDED LINEAR OPERATORS

Let  $X$  be a real normed space. We say that  $X$  is complete if and only if:

(Def. 16) For every sequence  $s_1$  of  $X$  such that  $s_1$  is Cauchy sequence by norm holds  $s_1$  is convergent.

Let us note that there exists a real normed space which is complete.

A real Banach space is a complete real normed space.

We now state three propositions:

(47) Let  $X$  be a real normed space and  $s_1$  be a sequence of  $X$ . If  $s_1$  is convergent, then  $\|s_1\|$  is convergent and  $\lim\|s_1\| = \|\lim s_1\|$ .

(48) Let  $X, Y$  be real normed spaces. Suppose  $Y$  is complete. Let  $s_1$  be a sequence of  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$ . If  $s_1$  is Cauchy sequence by norm, then  $s_1$  is convergent.

(49) For every real normed space  $X$  and for every real Banach space  $Y$  holds  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$  is a real Banach space.

Let  $X$  be a real normed space and let  $Y$  be a real Banach space. Observe that  $\text{RNormSpaceOfBoundedLinearOperators}(X, Y)$  is complete.

## REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [2] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(2):281–290, 1990.
- [3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [7] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in  $\mathcal{E}^2$ . *Formalized Mathematics*, 6(3):427–440, 1997.
- [8] Noboru Endou, Yasumasa Suzuki, and Yasunari Shidama. Real linear space of real sequences. *Formalized Mathematics*, 11(3):249–253, 2003.
- [9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.
- [10] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(3):477–481, 1990.
- [11] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.
- [12] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(3):471–475, 1990.
- [13] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(2):269–272, 1990.
- [14] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(2):335–342, 1990.
- [15] Henryk Orszczyżyn and Krzysztof Prażmowski. Real functions spaces. *Formalized Mathematics*, 1(3):555–561, 1990.
- [16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.

- [17] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(2):263–264, 1990.
- [18] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(4):511–516, 1991.
- [19] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(1):111–115, 1991.
- [20] Yasumasa Suzuki, Noboru Endou, and Yasunari Shidama. Banach space of absolute summable real sequences. *Formalized Mathematics*, 11(4):377–380, 2003.
- [21] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
- [22] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(2):329–334, 1990.
- [23] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(1):115–122, 1990.
- [24] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(1):25–34, 1990.
- [25] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(3):495–500, 1990.
- [26] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(1):9–11, 1990.
- [27] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(1):97–105, 1990.
- [28] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(4):341–347, 2003.
- [29] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(2):297–301, 1990.
- [30] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(2):291–296, 1990.
- [31] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [32] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [33] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.

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