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Complex Banach Space of Bounded Linear Operators

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Summary. An extension of [19]. In this article, the basic properties of complex linear spaces which are defined by the set of all complex linear operators from one complex linear space to another are described. Finally, a complex Banach space is introduced. This is defined by the set of all bounded complex linear operators, like in [19].

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The articles [24], [6], [26], [27], [4], [5], [17], [22], [21], [2], [1], [20], [11], [7], [25], [23], [18], [15], [13], [14], [12], [16], [3], [9], [10], [8], and [19] provide the terminology and notation for this paper.

1. Complex Vector Space of Operators

Let X be a set, let Y be a non empty set, let F be a function from $[: \mathbb{C}, Y :]$ into Y, let c be a complex number, and let f be a function from X into Y. Then $F^{\circ}(c, f)$ is an element of Y^X.

We now state the proposition

(1) Let X be a non empty set and Y be a complex linear space. Then there exists a function M_1 from $[: \mathbb{C}, (\text{the carrier of } Y)^X :]$ into (the carrier of $Y)^X$ such that for every Complex c and for every element f of (the carrier of $Y)^X$ and for every element s of X holds $M_1(\langle c, f \rangle)(s) = c \cdot f(s)$.

Let X be a non empty set and let Y be a complex linear space. The functor FuncExtMult(X, Y) yields a function from $[: \mathbb{C}, (\text{the carrier of } Y)^X :]$ into (the carrier of $Y)^X$ and is defined by the condition (Def. 1).

(Def. 1) Let c be a Complex, f be an element of (the carrier of $Y)^X$, and x be an element of X. Then $(\text{FuncExtMult}(X, Y))(\langle c, f \rangle)(x) = c \cdot f(x)$.

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We follow the rules: X is a non empty set, Y is a complex linear space, and f, g, h are elements of (the carrier of $Y)^X$.

We now state the proposition

(2) For every element x of X holds $(\mathrm{FuncZero}(X,Y))(x) = 0_Y$.

In the sequel a, b are Complexes.

Next we state several propositions:

(3) $h = (\mathrm{FuncExtMult}(X,Y))(\langle a, f\rangle)$ iff for every element x of X holds $h(x) = a \cdot f(x)$.

(4) $(\mathrm{FuncAdd}(X,Y))(f, g) = (\mathrm{FuncAdd}(X,Y))(g, f)$.

(5) $(\mathrm{FuncAdd}(X,Y))(f, (\mathrm{FuncAdd}(X,Y))(g, h)) = (\mathrm{FuncAdd}(X,Y))((\mathrm{FuncAdd}(X,Y))(f, g), h)$.

(6) $(\mathrm{FuncAdd}(X,Y))(\mathrm{FuncZero}(X,Y), f) = f$.

(7) $(\mathrm{FuncAdd}(X,Y))(f, (\mathrm{FuncExtMult}(X,Y))(\langle -1_{\mathbb{C}}, f\rangle)) = \mathrm{FuncZero}(X,Y)$.

(8) $(\mathrm{FuncExtMult}(X,Y))(\langle 1_{\mathbb{C}}, f\rangle) = f$.

(9) $(\mathrm{FuncExtMult}(X,Y))(\langle a, (\mathrm{FuncExtMult}(X,Y))(\langle b, f\rangle)\rangle) = (\mathrm{FuncExtMult}(X,Y))(\langle a \cdot b, f\rangle)$.

(10) $(\mathrm{FuncAdd}(X,Y))((\mathrm{FuncExtMult}(X,Y))(\langle a, f\rangle), (\mathrm{FuncExtMult}(X,Y))(\langle b, f\rangle)) = (\mathrm{FuncExtMult}(X,Y))(\langle a+b, f\rangle)$.

(11) $\langle$(the carrier of $Y)^X, \mathrm{FuncZero}(X,Y), \mathrm{FuncAdd}(X,Y), \mathrm{FuncExtMult}(X,Y)\rangle$ is a complex linear space.

Let X be a non empty set and let Y be a complex linear space. The functor ComplexVectSpace(X,Y) yielding a complex linear space is defined as follows:

(Def. 2) $\mathrm{ComplexVectSpace}(X,Y) = \langle$(the carrier of $Y)^X, \mathrm{FuncZero}(X,Y), \mathrm{FuncAdd}(X,Y), \mathrm{FuncExtMult}(X,Y)\rangle$.

Let X be a non empty set and let Y be a complex linear space. Observe that ComplexVectSpace(X,Y) is strict.

Let X be a non empty set and let Y be a complex linear space. Observe that every vector of ComplexVectSpace(X,Y) is function-like and relation-like.

Let X be a non empty set, let Y be a complex linear space, let f be a vector of ComplexVectSpace(X,Y), and let x be an element of X. Then $f(x)$ is a vector of Y.

We now state three propositions:

(12) Let X be a non empty set, Y be a complex linear space, and f, g, h be vectors of ComplexVectSpace(X,Y). Then $h = f + g$ if and only if for every element x of X holds $h(x) = f(x) + g(x)$.

(13) Let X be a non empty set, Y be a complex linear space, f, h be vectors of ComplexVectSpace(X,Y), and c be a Complex. Then $h = c \cdot f$ if and only if for every element x of X holds $h(x) = c \cdot f(x)$.

(14) For every non empty set X and for every complex linear space Y holds $0_{\mathrm{ComplexVectSpace}(X,Y)} = X \longmapsto 0_Y$.

2. Complex Vector Space of Linear Operators

Let X be a non empty CLS structure, let Y be a non empty loop structure, and let I_1 be a function from X into Y. We say that I_1 is additive if and only if:

(Def. 3) For all vectors x, y of X holds $I_1(x+y) = I_1(x) + I_1(y)$.

Let X, Y be non empty CLS structures and let I_1 be a function from X into Y. We say that I_1 is homogeneous if and only if:

(Def. 4) For every vector x of X and for every Complex r holds $I_1(r \cdot x) = r \cdot I_1(x)$.

Let X be a non empty CLS structure and let Y be a complex linear space. One can verify that there exists a function from X into Y which is additive and homogeneous.

Let X, Y be complex linear spaces. A linear operator from X into Y is an additive homogeneous function from X into Y.

Let X, Y be complex linear spaces. The functor LinearOperators(X,Y) yielding a subset of ComplexVectSpace(the carrier of X, Y) is defined by:

(Def. 5) For every set x holds $x \in$ LinearOperators(X,Y) iff x is a linear operator from X into Y.

Let X, Y be complex linear spaces. Note that LinearOperators(X,Y) is non empty.

Next we state two propositions:

(15) For all complex linear spaces X, Y holds LinearOperators(X,Y) is linearly closed.

(16) Let X, Y be complex linear spaces. Then $\langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), ComplexVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), ComplexVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), ComplexVectSpace(the carrier of X, Y))$\rangle$ is a subspace of ComplexVectSpace(the carrier of X, Y).

Let X, Y be complex linear spaces. One can check that $\langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), ComplexVectSpace(the carrier of X, Y)), Add_(LinearOperators(X,Y), ComplexVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X,Y), ComplexVectSpace(the carrier of X, Y))$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

Next we state the proposition

(17) Let X, Y be complex linear spaces. Then $\langle$LinearOperators(X,Y), Zero_(LinearOperators(X,Y), ComplexVectSpace(the carrier of X, Y)),

Add_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y))⟩ is a complex linear space.

Let X, Y be complex linear spaces. The functor CVSpLinOps(X, Y) yielding a complex linear space is defined as follows:

(Def. 6) CVSpLinOps(X, Y) = ⟨LinearOperators(X, Y), Zero_(LinearOperators (X, Y), ComplexVectSpace(the carrier of X, Y)), Add_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y)), Mult_(LinearOperators(X, Y), ComplexVectSpace(the carrier of X, Y))⟩.

Let X, Y be complex linear spaces. Note that CVSpLinOps(X, Y) is strict.

Let X, Y be complex linear spaces. One can check that every element of CVSpLinOps(X, Y) is function-like and relation-like.

Let X, Y be complex linear spaces, let f be an element of CVSpLinOps(X, Y), and let v be a vector of X. Then $f(v)$ is a vector of Y.

Next we state four propositions:

(18) Let X, Y be complex linear spaces and f, g, h be vectors of CVSpLinOps(X, Y). Then $h = f + g$ if and only if for every vector x of X holds $h(x) = f(x) + g(x)$.

(19) Let X, Y be complex linear spaces, f, h be vectors of CVSpLinOps(X, Y), and c be a Complex. Then $h = c \cdot f$ if and only if for every vector x of X holds $h(x) = c \cdot f(x)$.

(20) For all complex linear spaces X, Y holds $0_{\mathrm{CVSpLinOps}(X,Y)} =$ (the carrier of X) $\longmapsto 0_Y$.

(21) For all complex linear spaces X, Y holds (the carrier of X) $\longmapsto 0_Y$ is a linear operator from X into Y.

3. Complex Normed Linear Space of Bounded Linear Operators

One can prove the following proposition

(22) Let X be a complex normed space, s_1 be a sequence of X, and g be a point of X. If s_1 is convergent and $\lim s_1 = g$, then $\|s_1\|$ is convergent and $\lim \|s_1\| = \|g\|$.

Let X, Y be complex normed spaces and let I_1 be a linear operator from X into Y. We say that I_1 is bounded if and only if:

(Def. 7) There exists a real number K such that $0 \leqslant K$ and for every vector x of X holds $\|I_1(x)\| \leqslant K \cdot \|x\|$.

We now state the proposition

(23) Let X, Y be complex normed spaces and f be a linear operator from X into Y. If for every vector x of X holds $f(x) = 0_Y$, then f is bounded.

Let X, Y be complex normed spaces. Observe that there exists a linear operator from X into Y which is bounded.

Let X, Y be complex normed spaces. The functor BdLinOps(X,Y) yielding a subset of CVSpLinOps(X,Y) is defined as follows:

(Def. 8) For every set x holds $x \in$ BdLinOps(X,Y) iff x is a bounded linear operator from X into Y.

Let X, Y be complex normed spaces. One can check that BdLinOps(X,Y) is non empty.

One can prove the following two propositions:

(24) For all complex normed spaces X, Y holds BdLinOps(X,Y) is linearly closed.

(25) For all complex normed spaces X, Y holds $\langle$BdLinOps(X,Y), Zero_(BdLinOps(X,Y), CVSpLinOps(X,Y)), Add_(BdLinOps(X,Y), CVSpLinOps(X,Y)), Mult_(BdLinOps(X,Y), CVSpLinOps(X,Y))$\rangle$ is a subspace of CVSpLinOps(X,Y).

Let X, Y be complex normed spaces. Observe that $\langle$BdLinOps(X,Y), Zero_(BdLinOps(X,Y), CVSpLinOps(X,Y)), Add_(BdLinOps(X,Y), CVSpLinOps(X,Y)), Mult_(BdLinOps(X,Y), CVSpLinOps(X,Y))$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

Next we state the proposition

(26) For all complex normed spaces X, Y holds $\langle$BdLinOps(X,Y), Zero_(BdLinOps(X,Y), CVSpLinOps(X,Y)), Add_(BdLinOps(X,Y), CVSpLinOps(X,Y)), Mult_(BdLinOps(X,Y), CVSpLinOps(X,Y))$\rangle$ is a complex linear space.

Let X, Y be complex normed spaces. The functor CVSpBdLinOps(X,Y) yielding a complex linear space is defined by:

(Def. 9) CVSpBdLinOps$(X,Y) = \langle$BdLinOps(X,Y), Zero_(BdLinOps(X,Y), CVSpLinOps(X,Y)), Add_(BdLinOps(X,Y), CVSpLinOps(X,Y)), Mult_(BdLinOps(X,Y), CVSpLinOps(X,Y))$\rangle$.

Let X, Y be complex normed spaces. One can check that CVSpBdLinOps(X,Y) is strict.

Let X, Y be complex normed spaces. Note that every element of CVSpBdLinOps(X,Y) is function-like and relation-like.

Let X, Y be complex normed spaces, let f be an element of CVSpBdLinOps(X,Y), and let v be a vector of X. Then $f(v)$ is a vector of Y.

One can prove the following propositions:

(27) Let X, Y be complex normed spaces and f, g, h be vectors of CVSpBdLinOps(X,Y). Then $h = f + g$ if and only if for every vector

x of X holds $h(x) = f(x) + g(x)$.

(28) Let X, Y be complex normed spaces, f, h be vectors of CVSpBdLinOps(X,Y), and c be a Complex. Then $h = c \cdot f$ if and only if for every vector x of X holds $h(x) = c \cdot f(x)$.

(29) For all complex normed spaces X, Y holds $0_{\mathrm{CVSpBdLinOps}(X,Y)} =$ (the carrier of X) $\longmapsto 0_Y$.

Let X, Y be complex normed spaces and let f be a set. Let us assume that $f \in \mathrm{BdLinOps}(X,Y)$. The functor modetrans$(f,X,Y)$ yields a bounded linear operator from X into Y and is defined as follows:

(Def. 10) modetrans$(f,X,Y) = f$.

Let X, Y be complex normed spaces and let u be a linear operator from X into Y. The functor PreNorms(u) yielding a non empty subset of $\mathbb{R}$ is defined as follows:

(Def. 11) PreNorms$(u) = \{\|u(t)\|; t$ ranges over vectors of X: $\|t\| \leqslant 1\}$.

We now state three propositions:

(30) Let X, Y be complex normed spaces and g be a bounded linear operator from X into Y. Then PreNorms(g) is non empty and upper bounded.

(31) Let X, Y be complex normed spaces and g be a linear operator from X into Y. Then g is bounded if and only if PreNorms(g) is upper bounded.

(32) Let X, Y be complex normed spaces. Then there exists a function N_1 from BdLinOps(X,Y) into $\mathbb{R}$ such that for every set f if $f \in$ BdLinOps(X,Y), then $N_1(f) = \sup \mathrm{PreNorms}(\mathrm{modetrans}(f,X,Y))$.

Let X, Y be complex normed spaces. The functor BdLinOpsNorm(X,Y) yields a function from BdLinOps(X,Y) into $\mathbb{R}$ and is defined by:

(Def. 12) For every set x such that $x \in \mathrm{BdLinOps}(X,Y)$ holds $(\mathrm{BdLinOpsNorm}(X,Y))(x) = \sup \mathrm{PreNorms}(\mathrm{modetrans}(x,X,Y))$.

We now state two propositions:

(33) For all complex normed spaces X, Y and for every bounded linear operator f from X into Y holds modetrans$(f,X,Y) = f$.

(34) For all complex normed spaces X, Y and for every bounded linear operator f from X into Y holds $(\mathrm{BdLinOpsNorm}(X,Y))(f) = \sup \mathrm{PreNorms}(f)$.

Let X, Y be complex normed spaces. The functor CNSpBdLinOps(X,Y) yields a non empty complex normed space structure and is defined by:

(Def. 13) $\mathrm{CNSpBdLinOps}(X,Y) = \langle \mathrm{BdLinOps}(X,Y), \mathrm{Zero\_}(\mathrm{BdLinOps}(X,Y), \mathrm{CVSpLinOps}(X,Y)), \mathrm{Add\_}(\mathrm{BdLinOps}(X,Y), \mathrm{CVSpLinOps}(X,Y)), \mathrm{Mult\_}(\mathrm{BdLinOps}(X,Y), \mathrm{CVSpLinOps}(X,Y)), \mathrm{BdLinOpsNorm}(X,Y)\rangle$.

The following four propositions are true:

(35) For all complex normed spaces X, Y holds (the carrier of X) $\longmapsto 0_Y = 0_{\mathrm{CNSpBdLinOps}(X,Y)}$.

(36) Let X, Y be complex normed spaces, f be a point of CNSpBdLinOps(X, Y), and g be a bounded linear operator from X into Y. If $g = f$, then for every vector t of X holds $\|g(t)\| \leqslant \|f\| \cdot \|t\|$.

(37) For all complex normed spaces X, Y and for every point f of CNSpBdLinOps(X, Y) holds $0 \leqslant \|f\|$.

(38) For all complex normed spaces X, Y and for every point f of CNSpBdLinOps(X, Y) such that $f = 0_{\mathrm{CNSpBdLinOps}(X,Y)}$ holds $0 = \|f\|$.

Let X, Y be complex normed spaces. One can check that every element of CNSpBdLinOps(X, Y) is function-like and relation-like.

Let X, Y be complex normed spaces, let f be an element of CNSpBdLinOps(X, Y), and let v be a vector of X. Then $f(v)$ is a vector of Y.

We now state several propositions:

(39) Let X, Y be complex normed spaces and f, g, h be points of CNSpBdLinOps(X, Y). Then $h = f + g$ if and only if for every vector x of X holds $h(x) = f(x) + g(x)$.

(40) Let X, Y be complex normed spaces, f, h be points of CNSpBdLinOps(X, Y), and c be a Complex. Then $h = c \cdot f$ if and only if for every vector x of X holds $h(x) = c \cdot f(x)$.

(41) Let X, Y be complex normed spaces, f, g be points of CNSpBdLinOps(X, Y), and c be a Complex. Then $\|f\| = 0$ iff $f = 0_{\mathrm{CNSpBdLinOps}(X,Y)}$ and $\|c \cdot f\| = |c| \cdot \|f\|$ and $\|f + g\| \leqslant \|f\| + \|g\|$.

(42) For all complex normed spaces X, Y holds CNSpBdLinOps(X, Y) is complex normed space-like.

(43) For all complex normed spaces X, Y holds CNSpBdLinOps(X, Y) is a complex normed space.

Let X, Y be complex normed spaces. Observe that CNSpBdLinOps(X, Y) is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following proposition

(44) Let X, Y be complex normed spaces and f, g, h be points of CNSpBdLinOps(X, Y). Then $h = f - g$ if and only if for every vector x of X holds $h(x) = f(x) - g(x)$.

4. Complex Banach Space of Bounded Linear Operators

Let X be a complex normed space. We say that X is complete if and only if:

(Def. 14) For every sequence s_1 of X such that s_1 is Cauchy sequence by norm holds s_1 is convergent.

Let us observe that there exists a complex normed space which is complete.

A complex Banach space is a complete complex normed space.

One can prove the following three propositions:

(45) Let X be a complex normed space and s_1 be a sequence of X. If s_1 is convergent, then $\|s_1\|$ is convergent and $\lim\|s_1\| = \|\lim s_1\|$.

(46) Let X, Y be complex normed spaces. Suppose Y is complete. Let s_1 be a sequence of CNSpBdLinOps(X, Y). If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(47) For every complex normed space X and for every complex Banach space Y holds CNSpBdLinOps(X, Y) is a complex Banach space.

Let X be a complex normed space and let Y be a complex Banach space. One can verify that CNSpBdLinOps(X, Y) is complete.

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