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Complex Linear Space and Complex Normed Space

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Summary. In this article, we introduce the notion of complex linear space and complex normed space.

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The articles [16], [7], [18], [1], [14], [13], [15], [8], [19], [4], [5], [2], [11], [17], [6], [10], [9], [3], and [12] provide the terminology and notation for this paper.

1. Complex Linear Space

We consider CLS structures as extensions of loop structure as systems
⟨ a carrier, a zero, an addition, an external multiplication ⟩,
where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, and the external multiplication is a function from $[:\mathbb{C}, \text{the carrier}:]$ into the carrier.

Let us observe that there exists a CLS structure which is non empty.

Let V be a CLS structure. A vector of V is an element of V.

Let V be a non empty CLS structure, let v be a vector of V, and let z be a Complex. The functor $z \cdot v$ yielding an element of V is defined as follows:

(Def. 1) $z \cdot v = (\text{the external multiplication of } V)(\langle z, v \rangle)$.

Let Z_1 be a non empty set, let O be an element of Z_1, let F be a binary operation on Z_1, and let G be a function from $[:\mathbb{C}, Z_1:]$ into Z_1. One can verify that $\langle Z_1, O, F, G \rangle$ is non empty.

Let I_1 be a non empty CLS structure. We say that I_1 is complex linear space-like if and only if the conditions (Def. 2) are satisfied.

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(Def. 2)(i) For every Complex z and for all vectors v, w of I_1 holds $z \cdot (v + w) = z \cdot v + z \cdot w$,

(ii) for all Complexes z_1, z_2 and for every vector v of I_1 holds $(z_1 + z_2) \cdot v = z_1 \cdot v + z_2 \cdot v$,

(iii) for all Complexes z_1, z_2 and for every vector v of I_1 holds $(z_1 \cdot z_2) \cdot v = z_1 \cdot (z_2 \cdot v)$, and

(iv) for every vector v of I_1 holds $1_{\mathbb{C}} \cdot v = v$.

Let us observe that there exists a non empty CLS structure which is non empty, strict, Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

A complex linear space is an Abelian add-associative right zeroed right complementable complex linear space-like non empty CLS structure.

One can prove the following proposition

(1) Let V be a non empty CLS structure. Suppose that for all vectors v, w of V holds $v + w = w + v$ and for all vectors u, v, w of V holds $(u+v)+w = u+(v+w)$ and for every vector v of V holds $v+0_V = v$ and for every vector v of V there exists a vector w of V such that $v + w = 0_V$ and for every Complex z and for all vectors v, w of V holds $z \cdot (v + w) = z \cdot v + z \cdot w$ and for all Complexes z_1, z_2 and for every vector v of V holds $(z_1 + z_2) \cdot v = z_1 \cdot v + z_2 \cdot v$ and for all Complexes z_1, z_2 and for every vector v of V holds $(z_1 \cdot z_2) \cdot v = z_1 \cdot (z_2 \cdot v)$ and for every vector v of V holds $1_{\mathbb{C}} \cdot v = v$. Then V is a complex linear space.

We adopt the following convention: V, X, Y are complex linear spaces, u, v, v_1, v_2 are vectors of V, and z, z_1, z_2 are Complexes.

The following propositions are true:

(2) If $z = 0_{\mathbb{C}}$ or $v = 0_V$, then $z \cdot v = 0_V$.

(3) If $z \cdot v = 0_V$, then $z = 0_{\mathbb{C}}$ or $v = 0_V$.

(4) $-v = (-1_{\mathbb{C}}) \cdot v$.

(5) If $v = -v$, then $v = 0_V$.

(6) If $v + v = 0_V$, then $v = 0_V$.

(7) $z \cdot -v = (-z) \cdot v$.

(8) $z \cdot -v = -z \cdot v$.

(9) $(-z) \cdot -v = z \cdot v$.

(10) $z \cdot (v - u) = z \cdot v - z \cdot u$.

(11) $(z_1 - z_2) \cdot v = z_1 \cdot v - z_2 \cdot v$.

(12) If $z \neq 0$ and $z \cdot v = z \cdot u$, then $v = u$.

(13) If $v \neq 0_V$ and $z_1 \cdot v = z_2 \cdot v$, then $z_1 = z_2$.

(14) Let F, G be finite sequences of elements of the carrier of V. Suppose $\operatorname{len} F = \operatorname{len} G$ and for every natural number k and for every vector v of V

such that $k \in \operatorname{dom} F$ and $v = G(k)$ holds $F(k) = z \cdot v$. Then $\sum F = z \cdot \sum G$.

(15) $z \cdot \sum(\varepsilon_{(\text{the carrier of } V)}) = 0_V$.

(16) $z \cdot \sum\langle v, u\rangle = z \cdot v + z \cdot u$.

(17) $z \cdot \sum\langle u, v_1, v_2\rangle = z \cdot u + z \cdot v_1 + z \cdot v_2$.

(18) $\sum\langle v, v\rangle = (2 + 0i) \cdot v$.

(19) $\sum\langle -v, -v\rangle = (-2 + 0i) \cdot v$.

(20) $\sum\langle v, v, v\rangle = (3 + 0i) \cdot v$.

2. Subspace and Cosets of Subspaces in Complex Linear Space

In the sequel V_1, V_2, V_3 are subsets of V.

Let us consider V, V_1. We say that V_1 is linearly closed if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) For all vectors v, u of V such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$, and

(ii) for every Complex z and for every vector v of V such that $v \in V_1$ holds $z \cdot v \in V_1$.

Next we state several propositions:

(21) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $0_V \in V_1$.

(22) If V_1 is linearly closed, then for every vector v of V such that $v \in V_1$ holds $-v \in V_1$.

(23) If V_1 is linearly closed, then for all vectors v, u of V such that $v \in V_1$ and $u \in V_1$ holds $v - u \in V_1$.

(24) $\{0_V\}$ is linearly closed.

(25) If the carrier of $V = V_1$, then V_1 is linearly closed.

(26) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v + u : v \in V_1 \wedge u \in V_2\}$, then V_3 is linearly closed.

(27) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider V. A complex linear space is said to be a subspace of V if it satisfies the conditions (Def. 4).

(Def. 4)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the zero of it = the zero of V,

(iii) the addition of it = (the addition of V)$\restriction[\!:$the carrier of it, the carrier of it$:\!]$, and

(iv) the external multiplication of it = (the external multiplication of V)$\restriction[\!:\mathbb{C}$, the carrier of it$:\!]$.

We use the following convention: W, W_1, W_2 denote subspaces of V, x denotes a set, and w, w_1, w_2 denote vectors of W.

We now state a number of propositions:

(28) If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(29) If $x \in W$, then $x \in V$.

(30) w is a vector of V.

(31) $0_W = 0_V$.

(32) $0_{(W_1)} = 0_{(W_2)}$.

(33) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(34) If $w = v$, then $z \cdot w = z \cdot v$.

(35) If $w = v$, then $-v = -w$.

(36) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(37) $0_V \in W$.

(38) $0_{(W_1)} \in W_2$.

(39) $0_W \in V$.

(40) If $u \in W$ and $v \in W$, then $u + v \in W$.

(41) If $v \in W$, then $z \cdot v \in W$.

(42) If $v \in W$, then $-v \in W$.

(43) If $u \in W$ and $v \in W$, then $u - v \in W$.

In the sequel D denotes a non empty set, d_1 denotes an element of D, A denotes a binary operation on D, and M denotes a function from $[\!: \mathbb{C}, D :\!]$ into D.

Next we state several propositions:

(44) Suppose $V_1 = D$ and $d_1 = 0_V$ and $A =$ (the addition of V)$\restriction[\!: V_1, V_1 :\!]$ and $M =$ (the external multiplication of V)$\restriction[\!: \mathbb{C}, V_1 :\!]$. Then $\langle D, d_1, A, M \rangle$ is a subspace of V.

(45) V is a subspace of V.

(46) Let V, X be strict complex linear spaces. If V is a subspace of X and X is a subspace of V, then $V = X$.

(47) If V is a subspace of X and X is a subspace of Y, then V is a subspace of Y.

(48) If the carrier of $W_1 \subseteq$ the carrier of W_2, then W_1 is a subspace of W_2.

(49) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a subspace of W_2.

Let us consider V. Observe that there exists a subspace of V which is strict.

The following propositions are true:

(50) For all strict subspaces W_1, W_2 of V such that the carrier of $W_1 =$ the carrier of W_2 holds $W_1 = W_2$.

(51) For all strict subspaces W_1, W_2 of V such that for every v holds $v \in W_1$ iff $v \in W_2$ holds $W_1 = W_2$.

(52) Let V be a strict complex linear space and W be a strict subspace of V. If the carrier of $W =$ the carrier of V, then $W = V$.

(53) Let V be a strict complex linear space and W be a strict subspace of V. If for every vector v of V holds $v \in W$ iff $v \in V$, then $W = V$.

(54) If the carrier of $W = V_1$, then V_1 is linearly closed.

(55) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists a strict subspace W of V such that $V_1 =$ the carrier of W.

Let us consider V. The functor $\mathbf{0}_V$ yields a strict subspace of V and is defined by:

(Def. 5) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let us consider V. The functor Ω_V yields a strict subspace of V and is defined as follows:

(Def. 6) $\Omega_V =$ the CLS structure of V.

We now state several propositions:

(56) $\mathbf{0}_W = \mathbf{0}_V$.

(57) $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(58) $\mathbf{0}_W$ is a subspace of V.

(59) $\mathbf{0}_V$ is a subspace of W.

(60) $\mathbf{0}_{(W_1)}$ is a subspace of W_2.

(61) Every strict complex linear space V is a subspace of Ω_V.

Let us consider V and let us consider v, W. The functor $v + W$ yielding a subset of V is defined by:

(Def. 7) $v + W = \{v + u : u \in W\}$.

Let us consider V and let us consider W. A subset of V is called a coset of W if:

(Def. 8) There exists v such that it $= v + W$.

In the sequel B, C denote cosets of W.

The following propositions are true:

(62) $0_V \in v + W$ iff $v \in W$.

(63) $v \in v + W$.

(64) $0_V + W =$ the carrier of W.

(65) $v + \mathbf{0}_V = \{v\}$.

(66) $v + \Omega_V =$ the carrier of V.

(67) $0_V \in v + W$ iff $v + W =$ the carrier of W.

(68) $v \in W$ iff $v + W =$ the carrier of W.

(69) If $v \in W$, then $z \cdot v + W =$ the carrier of W.

(70) If $z \neq 0_{\mathbb{C}}$ and $z \cdot v + W =$ the carrier of W, then $v \in W$.

(71) $v \in W$ iff $-v + W =$ the carrier of W.

(72) $u \in W$ iff $v + W = v + u + W$.

(73) $u \in W$ iff $v + W = (v - u) + W$.

(74) $v \in u + W$ iff $u + W = v + W$.

(75) $v + W = -v + W$ iff $v \in W$.

(76) If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.

(77) If $u \in v + W$ and $u \in -v + W$, then $v \in W$.

(78) If $z \neq 1_{\mathbb{C}}$ and $z \cdot v \in v + W$, then $v \in W$.

(79) If $v \in W$, then $z \cdot v \in v + W$.

(80) $-v \in v + W$ iff $v \in W$.

(81) $u + v \in v + W$ iff $u \in W$.

(82) $v - u \in v + W$ iff $u \in W$.

(83) $u \in v + W$ iff there exists v_1 such that $v_1 \in W$ and $u = v + v_1$.

(84) $u \in v + W$ iff there exists v_1 such that $v_1 \in W$ and $u = v - v_1$.

(85) There exists v such that $v_1 \in v + W$ and $v_2 \in v + W$ iff $v_1 - v_2 \in W$.

(86) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v + v_1 = u$.

(87) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v - v_1 = u$.

(88) For all strict subspaces W_1, W_2 of V holds $v + W_1 = v + W_2$ iff $W_1 = W_2$.

(89) For all strict subspaces W_1, W_2 of V such that $v + W_1 = u + W_2$ holds $W_1 = W_2$.

(90) C is linearly closed iff $C =$ the carrier of W.

(91) For all strict subspaces W_1, W_2 of V and for every coset C_1 of W_1 and for every coset C_2 of W_2 such that $C_1 = C_2$ holds $W_1 = W_2$.

(92) $\{v\}$ is a coset of $\mathbf{0}_V$.

(93) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.

(94) The carrier of W is a coset of W.

(95) The carrier of V is a coset of Ω_V.

(96) If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of V.

(97) $0_V \in C$ iff $C =$ the carrier of W.

(98) $u \in C$ iff $C = u + W$.

(99) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u + v_1 = v$.

(100) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u - v_1 = v$.

(101) There exists C such that $v_1 \in C$ and $v_2 \in C$ iff $v_1 - v_2 \in W$.

(102) If $u \in B$ and $u \in C$, then $B = C$.

3. Complex Normed Space

We consider complex normed space structures as extensions of CLS structure as systems

⟨ a carrier, a zero, an addition, an external multiplication, a norm ⟩,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from $[:\mathbb{C}, \text{the carrier}:]$ into the carrier, and the norm is a function from the carrier into $\mathbb{R}$.

Let us mention that there exists a complex normed space structure which is non empty.

In the sequel X is a non empty complex normed space structure and x is a point of X.

Let us consider X, x. The functor $\|x\|$ yielding a real number is defined by:

(Def. 9) $\|x\| = (\text{the norm of } X)(x)$.

Let I_1 be a non empty complex normed space structure. We say that I_1 is complex normed space-like if and only if:

(Def. 10) For all points x, y of I_1 and for every z holds $\|x\| = 0$ iff $x = 0_{(I_1)}$ and $\|z \cdot x\| = |z| \cdot \|x\|$ and $\|x + y\| \leqslant \|x\| + \|y\|$.

One can verify that there exists a non empty complex normed space structure which is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, right complementable, and strict.

A complex normed space is a complex normed space-like complex linear space-like Abelian add-associative right zeroed right complementable non empty complex normed space structure.

We follow the rules: C_3 is a complex normed space and x, y, w, g are points of C_3.

The following propositions are true:

(103) $\|0_{(C_3)}\| = 0$.

(104) $\|-x\| = \|x\|$.

(105) $\|x - y\| \leqslant \|x\| + \|y\|$.

(106) $0 \leqslant \|x\|$.

(107) $\|z_1 \cdot x + z_2 \cdot y\| \leqslant |z_1| \cdot \|x\| + |z_2| \cdot \|y\|$.

(108) $\|x - y\| = 0$ iff $x = y$.

(109) $\|x - y\| = \|y - x\|$.

(110) $\|x\| - \|y\| \leqslant \|x - y\|$.

(111) $|\|x\| - \|y\|| \leqslant \|x - y\|$.

(112) $\|x - w\| \leqslant \|x - y\| + \|y - w\|$.

(113) If $x \neq y$, then $\|x - y\| \neq 0$.

We adopt the following rules: S, S_1, S_2 are sequences of C_3, n, m are natural numbers, and r is a real number.

One can prove the following proposition

(114) There exists S such that $\operatorname{rng} S = \{0_{(C_3)}\}$.

In this article we present several logical schemes. The scheme *ExCNSSeq* deals with a complex normed space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExCLSSeq* deals with a complex linear space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

Let C_3 be a complex linear space and let S_1, S_2 be sequences of C_3. The functor $S_1 + S_2$ yielding a sequence of C_3 is defined by:

(Def. 11) For every n holds $(S_1 + S_2)(n) = S_1(n) + S_2(n)$.

Let C_3 be a complex linear space and let S_1, S_2 be sequences of C_3. The functor $S_1 - S_2$ yielding a sequence of C_3 is defined by:

(Def. 12) For every n holds $(S_1 - S_2)(n) = S_1(n) - S_2(n)$.

Let C_3 be a complex linear space, let S be a sequence of C_3, and let x be an element of C_3. The functor $S - x$ yielding a sequence of C_3 is defined by:

(Def. 13) For every n holds $(S - x)(n) = S(n) - x$.

Let C_3 be a complex linear space, let S be a sequence of C_3, and let us consider z. The functor $z \cdot S$ yields a sequence of C_3 and is defined as follows:

(Def. 14) For every n holds $(z \cdot S)(n) = z \cdot S(n)$.

Let us consider C_3 and let us consider S. We say that S is convergent if and only if:

(Def. 15) There exists g such that for every r such that $0 < r$ there exists m such that for every n such that $m \leqslant n$ holds $\|S(n) - g\| < r$.

The following four propositions are true:

(115) If S_1 is convergent and S_2 is convergent, then $S_1 + S_2$ is convergent.

(116) If S_1 is convergent and S_2 is convergent, then $S_1 - S_2$ is convergent.

(117) If S is convergent, then $S - x$ is convergent.

(118) If S is convergent, then $z \cdot S$ is convergent.

Let us consider C_3 and let us consider S. The functor $\|S\|$ yielding a sequence of real numbers is defined as follows:

(Def. 16) For every n holds $\|S\|(n) = \|S(n)\|$.

The following proposition is true

(119) If S is convergent, then $\|S\|$ is convergent.

Let us consider C_3 and let us consider S. Let us assume that S is convergent. The functor $\lim S$ yields a point of C_3 and is defined as follows:

(Def. 17) For every r such that $0 < r$ there exists m such that for every n such that $m \leqslant n$ holds $\|S(n) - \lim S\| < r$.

The following propositions are true:

(120) If S is convergent and $\lim S = g$, then $\|S - g\|$ is convergent and $\lim\|S - g\| = 0$.

(121) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 + S_2) = \lim S_1 + \lim S_2$.

(122) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 - S_2) = \lim S_1 - \lim S_2$.

(123) If S is convergent, then $\lim(S - x) = \lim S - x$.

(124) If S is convergent, then $\lim(z \cdot S) = z \cdot \lim S$.

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