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Convergent Sequences in Complex Unitary Space

Noboru Endou
Gifu National College of Technology

Summary. In this article, we introduce the notion of convergence sequence in complex unitary space and complex Hilbert space.

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The terminology and notation used in this paper are introduced in the following papers: [15], [2], [14], [7], [1], [17], [3], [4], [10], [9], [16], [13], [11], [12], [8], [5], and [6].

1. Convergence in Complex Unitary Space

For simplicity, we adopt the following convention: X is a complex unitary space, x, y, w, g, g_1, g_2 are points of X, z is a Complex, q, r, M are real numbers, s_1, s_2, s_3, s_4 are sequences of X, k, n, m are natural numbers, and N_1 is an increasing sequence of naturals.

Let us consider X, s_1. We say that s_1 is convergent if and only if:

(Def. 1) There exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\rho(s_1(n), g) < r$.

Next we state several propositions:

(1) If s_1 is constant, then s_1 is convergent.

(2) If s_2 is convergent and there exists k such that for every n such that $k \leqslant n$ holds $s_3(n) = s_2(n)$, then s_3 is convergent.

(3) If s_2 is convergent and s_3 is convergent, then $s_2 + s_3$ is convergent.

(4) If s_2 is convergent and s_3 is convergent, then $s_2 - s_3$ is convergent.

(5) If s_1 is convergent, then $z \cdot s_1$ is convergent.

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(6) If s_1 is convergent, then $-s_1$ is convergent.

(7) If s_1 is convergent, then $s_1 + x$ is convergent.

(8) If s_1 is convergent, then $s_1 - x$ is convergent.

(9) s_1 is convergent if and only if there exists g such that for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields a point of X and is defined as follows:

(Def. 2) For every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\rho(s_1(n), \lim s_1) < r$.

One can prove the following propositions:

(10) If s_1 is constant and $x \in \operatorname{rng} s_1$, then $\lim s_1 = x$.

(11) If s_1 is constant and there exists n such that $s_1(n) = x$, then $\lim s_1 = x$.

(12) If s_2 is convergent and there exists k such that for every n such that $n \geqslant k$ holds $s_3(n) = s_2(n)$, then $\lim s_2 = \lim s_3$.

(13) If s_2 is convergent and s_3 is convergent, then $\lim(s_2+s_3) = \lim s_2+\lim s_3$.

(14) If s_2 is convergent and s_3 is convergent, then $\lim(s_2-s_3) = \lim s_2-\lim s_3$.

(15) If s_1 is convergent, then $\lim(z \cdot s_1) = z \cdot \lim s_1$.

(16) If s_1 is convergent, then $\lim(-s_1) = -\lim s_1$.

(17) If s_1 is convergent, then $\lim(s_1 + x) = \lim s_1 + x$.

(18) If s_1 is convergent, then $\lim(s_1 - x) = \lim s_1 - x$.

(19) Suppose s_1 is convergent. Then $\lim s_1 = g$ if and only if for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\|s_1(n) - g\| < r$.

Let us consider X, s_1. The functor $\|s_1\|$ yielding a sequence of real numbers is defined as follows:

(Def. 3) For every n holds $\|s_1\|(n) = \|s_1(n)\|$.

One can prove the following three propositions:

(20) If s_1 is convergent, then $\|s_1\|$ is convergent.

(21) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1\|$ is convergent and $\lim\|s_1\| = \|g\|$.

(22) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - g\|$ is convergent and $\lim\|s_1 - g\| = 0$.

Let us consider X, s_1, x. The functor $\rho(s_1, x)$ yielding a sequence of real numbers is defined as follows:

(Def. 4) For every n holds $(\rho(s_1, x))(n) = \rho(s_1(n), x)$.

One can prove the following propositions:

(23) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent.

(24) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1, g)$ is convergent and $\lim \rho(s_1, g) = 0$.

(25) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 + s_3\|$ is convergent and $\lim \|s_2 + s_3\| = \|g_1 + g_2\|$.

(26) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|(s_2+s_3)-(g_1+g_2)\|$ is convergent and $\lim \|(s_2+s_3)-(g_1+g_2)\| = 0$.

(27) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3\|$ is convergent and $\lim \|s_2 - s_3\| = \|g_1 - g_2\|$.

(28) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\|s_2 - s_3 - (g_1 - g_2)\|$ is convergent and $\lim \|s_2 - s_3 - (g_1 - g_2)\| = 0$.

(29) If s_1 is convergent and $\lim s_1 = g$, then $\|z \cdot s_1\|$ is convergent and $\lim \|z \cdot s_1\| = \|z \cdot g\|$.

(30) If s_1 is convergent and $\lim s_1 = g$, then $\|z \cdot s_1 - z \cdot g\|$ is convergent and $\lim \|z \cdot s_1 - z \cdot g\| = 0$.

(31) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1\|$ is convergent and $\lim \|-s_1\| = \|-g\|$.

(32) If s_1 is convergent and $\lim s_1 = g$, then $\|-s_1 - -g\|$ is convergent and $\lim \|-s_1 - -g\| = 0$.

(33) If s_1 is convergent and $\lim s_1 = g$, then $\|(s_1+x)-(g+x)\|$ is convergent and $\lim \|(s_1+x)-(g+x)\| = 0$.

(34) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x\|$ is convergent and $\lim \|s_1 - x\| = \|g - x\|$.

(35) If s_1 is convergent and $\lim s_1 = g$, then $\|s_1 - x - (g - x)\|$ is convergent and $\lim \|s_1 - x - (g - x)\| = 0$.

(36) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 + s_3, g_1 + g_2)$ is convergent and $\lim \rho(s_2 + s_3, g_1 + g_2) = 0$.

(37) If s_2 is convergent and $\lim s_2 = g_1$ and s_3 is convergent and $\lim s_3 = g_2$, then $\rho(s_2 - s_3, g_1 - g_2)$ is convergent and $\lim \rho(s_2 - s_3, g_1 - g_2) = 0$.

(38) If s_1 is convergent and $\lim s_1 = g$, then $\rho(z \cdot s_1, z \cdot g)$ is convergent and $\lim \rho(z \cdot s_1, z \cdot g) = 0$.

(39) If s_1 is convergent and $\lim s_1 = g$, then $\rho(s_1+x, g+x)$ is convergent and $\lim \rho(s_1 + x, g + x) = 0$.

Let us consider X, x, r. The functor $\mathrm{Ball}(x, r)$ yields a subset of X and is defined by:

(Def. 5) $\mathrm{Ball}(x, r) = \{y; y$ ranges over points of X: $\|x - y\| < r\}$.

The functor $\overline{\mathrm{Ball}}(x, r)$ yielding a subset of X is defined by:

(Def. 6) $\overline{\mathrm{Ball}}(x, r) = \{y; y$ ranges over points of X: $\|x - y\| \leqslant r\}$.

The functor $\mathrm{Sphere}(x, r)$ yielding a subset of X is defined as follows:

(Def. 7) $\mathrm{Sphere}(x, r) = \{y; y$ ranges over points of X: $\|x - y\| = r\}$.

Next we state a number of propositions:

(40) $w \in \mathrm{Ball}(x, r)$ iff $\|x - w\| < r$.

(41) $w \in \mathrm{Ball}(x, r)$ iff $\rho(x, w) < r$.

(42) If $r > 0$, then $x \in \mathrm{Ball}(x, r)$.

(43) If $y \in \mathrm{Ball}(x, r)$ and $w \in \mathrm{Ball}(x, r)$, then $\rho(y, w) < 2 \cdot r$.

(44) If $y \in \mathrm{Ball}(x, r)$, then $y - w \in \mathrm{Ball}(x - w, r)$.

(45) If $y \in \mathrm{Ball}(x, r)$, then $y - x \in \mathrm{Ball}(0_X, r)$.

(46) If $y \in \mathrm{Ball}(x, r)$ and $r \leqslant q$, then $y \in \mathrm{Ball}(x, q)$.

(47) $w \in \overline{\mathrm{Ball}}(x, r)$ iff $\|x - w\| \leqslant r$.

(48) $w \in \overline{\mathrm{Ball}}(x, r)$ iff $\rho(x, w) \leqslant r$.

(49) If $r \geqslant 0$, then $x \in \overline{\mathrm{Ball}}(x, r)$.

(50) If $y \in \mathrm{Ball}(x, r)$, then $y \in \overline{\mathrm{Ball}}(x, r)$.

(51) $w \in \mathrm{Sphere}(x, r)$ iff $\|x - w\| = r$.

(52) $w \in \mathrm{Sphere}(x, r)$ iff $\rho(x, w) = r$.

(53) If $y \in \mathrm{Sphere}(x, r)$, then $y \in \overline{\mathrm{Ball}}(x, r)$.

(54) $\mathrm{Ball}(x, r) \subseteq \overline{\mathrm{Ball}}(x, r)$.

(55) $\mathrm{Sphere}(x, r) \subseteq \overline{\mathrm{Ball}}(x, r)$.

(56) $\mathrm{Ball}(x, r) \cup \mathrm{Sphere}(x, r) = \overline{\mathrm{Ball}}(x, r)$.

2. Cauchy Sequence and Hilbert Space with Complex Coefficient

Let us consider X and let us consider s_1. We say that s_1 is Cauchy if and only if:

(Def. 8) For every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\rho(s_1(n), s_1(m)) < r$.

The following propositions are true:

(57) If s_1 is constant, then s_1 is Cauchy.

(58) s_1 is Cauchy if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|s_1(n) - s_1(m)\| < r$.

(59) If s_2 is Cauchy and s_3 is Cauchy, then $s_2 + s_3$ is Cauchy.

(60) If s_2 is Cauchy and s_3 is Cauchy, then $s_2 - s_3$ is Cauchy.

(61) If s_1 is Cauchy, then $z \cdot s_1$ is Cauchy.

(62) If s_1 is Cauchy, then $-s_1$ is Cauchy.

(63) If s_1 is Cauchy, then $s_1 + x$ is Cauchy.

(64) If s_1 is Cauchy, then $s_1 - x$ is Cauchy.

(65) If s_1 is convergent, then s_1 is Cauchy.

Let us consider X and let us consider s_2, s_3. We say that s_2 is compared to s_3 if and only if:

(Def. 9) For every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\rho(s_2(n), s_3(n)) < r$.

One can prove the following two propositions:

(66) s_1 is compared to s_1.

(67) If s_2 is compared to s_3, then s_3 is compared to s_2.

Let us consider X and let us consider s_2, s_3. Let us notice that the predicate s_2 is compared to s_3 is reflexive and symmetric.

The following propositions are true:

(68) If s_2 is compared to s_3 and s_3 is compared to s_4, then s_2 is compared to s_4.

(69) s_2 is compared to s_3 iff for every r such that $r > 0$ there exists m such that for every n such that $n \geqslant m$ holds $\|s_2(n) - s_3(n)\| < r$.

(70) If there exists k such that for every n such that $n \geqslant k$ holds $s_2(n) = s_3(n)$, then s_2 is compared to s_3.

(71) If s_2 is Cauchy and compared to s_3, then s_3 is Cauchy.

(72) If s_2 is convergent and compared to s_3, then s_3 is convergent.

(73) If s_2 is convergent and $\lim s_2 = g$ and s_2 is compared to s_3, then s_3 is convergent and $\lim s_3 = g$.

Let us consider X and let us consider s_1. We say that s_1 is bounded if and only if:

(Def. 10) There exists M such that $M > 0$ and for every n holds $\|s_1(n)\| \leqslant M$.

We now state several propositions:

(74) If s_2 is bounded and s_3 is bounded, then $s_2 + s_3$ is bounded.

(75) If s_1 is bounded, then $-s_1$ is bounded.

(76) If s_2 is bounded and s_3 is bounded, then $s_2 - s_3$ is bounded.

(77) If s_1 is bounded, then $z \cdot s_1$ is bounded.

(78) If s_1 is constant, then s_1 is bounded.

(79) For every m there exists M such that $M > 0$ and for every n such that $n \leqslant m$ holds $\|s_1(n)\| < M$.

(80) If s_1 is convergent, then s_1 is bounded.

(81) If s_2 is bounded and compared to s_3, then s_3 is bounded.

Let us consider X, N_1, s_1. Then $s_1 \cdot N_1$ is a sequence of X.

We now state several propositions:

(82) Let X be a complex unitary space, s be a sequence of X, N be an increasing sequence of naturals, and n be a natural number. Then $(s \cdot N)(n) = s(N(n))$.

(83) s_1 is a subsequence of s_1.

(84) If s_2 is a subsequence of s_3 and s_3 is a subsequence of s_4, then s_2 is a subsequence of s_4.

(85) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(86) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

(87) If s_1 is bounded and s_2 is a subsequence of s_1, then s_2 is bounded.

(88) If s_1 is convergent and s_2 is a subsequence of s_1, then s_2 is convergent.

(89) If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(90) If s_1 is Cauchy and s_2 is a subsequence of s_1, then s_2 is Cauchy.

Let us consider X, let us consider s_1, and let us consider k. The functor $s_1 \uparrow k$ yields a sequence of X and is defined as follows:

(Def. 11) For every n holds $(s_1 \uparrow k)(n) = s_1(n + k)$.

One can prove the following propositions:

(91) $s_1 \uparrow 0 = s_1$.

(92) $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(93) $s_1 \uparrow k \uparrow m = s_1 \uparrow (k + m)$.

(94) $(s_2 + s_3) \uparrow k = s_2 \uparrow k + s_3 \uparrow k$.

(95) $(-s_1) \uparrow k = -s_1 \uparrow k$.

(96) $(s_2 - s_3) \uparrow k = s_2 \uparrow k - s_3 \uparrow k$.

(97) $(z \cdot s_1) \uparrow k = z \cdot (s_1 \uparrow k)$.

(98) $(s_1 \cdot N_1) \uparrow k = s_1 \cdot (N_1 \uparrow k)$.

(99) $s_1 \uparrow k$ is a subsequence of s_1.

(100) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(101) If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is convergent.

(102) If s_1 is Cauchy and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is Cauchy.

(103) If s_1 is Cauchy, then $s_1 \uparrow k$ is Cauchy.

(104) If s_2 is compared to s_3, then $s_2 \uparrow k$ is compared to $s_3 \uparrow k$.

(105) If s_1 is bounded, then $s_1 \uparrow k$ is bounded.

(106) If s_1 is constant, then $s_1 \uparrow k$ is constant.

Let us consider X. We say that X is complete if and only if:

(Def. 12) For every s_1 such that s_1 is Cauchy holds s_1 is convergent.

The following proposition is true

(107) If X is complete and s_1 is Cauchy, then s_1 is bounded.

Let us consider X. We say that X is Hilbert if and only if:

(Def. 13) X is a complex unitary space and complete.

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