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Hilbert Space of Complex Sequences

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Summary. An extension of [9]. As the example of complex norm spaces, we introduce the arithmetic addition and multiplication in the set of absolute summable complex sequences and also introduce the norm.

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The papers [18], [21], [5], [17], [10], [22], [3], [4], [20], [19], [13], [11], [12], [15], [2], [1], [14], [16], [6], [8], and [7] provide the notation and terminology for this paper.

1. Hilbert Space of Complex Sequences

One can prove the following propositions:

(1) The carrier of Complexl2-Space = the set of l2-complex sequences and for every set x holds x is an element of Complexl2-Space iff x is a complex sequence and $|\mathrm{id}_{\mathrm{seq}}(x)|\,|\mathrm{id}_{\mathrm{seq}}(x)|$ is summable and for every set x holds x is an element of Complexl2-Space iff x is a complex sequence and $\mathrm{id}_{\mathrm{seq}}(x)\,\overline{\mathrm{id}_{\mathrm{seq}}(x)}$ is absolutely summable and $0_{\mathrm{Complexl2\text{-}Space}} = \mathrm{CZeroseq}$ and for every vector u of Complexl2-Space holds $u = \mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of Complexl2-Space holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$ and for every Complex r and for every vector u of Complexl2-Space holds $r \cdot u = r\ \mathrm{id}_{\mathrm{seq}}(u)$ and for every vector u of Complexl2-Space holds $-u = -\mathrm{id}_{\mathrm{seq}}(u)$ and $\mathrm{id}_{\mathrm{seq}}(-u) = -\mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of Complexl2-Space holds $u - v = \mathrm{id}_{\mathrm{seq}}(u) - \mathrm{id}_{\mathrm{seq}}(v)$ and for all vectors v, w of Complexl2-Space holds $|\mathrm{id}_{\mathrm{seq}}(v)|\,|\mathrm{id}_{\mathrm{seq}}(w)|$ is summable and for all vectors v, w of Complexl2-Space holds $(v|w) = \sum(\mathrm{id}_{\mathrm{seq}}(v)\ \overline{\mathrm{id}_{\mathrm{seq}}(w)})$.

(2) Let x, y, z be points of Complexl2-Space and a be a Complex. Then $(x|x) = 0$ iff $x = 0_{\mathrm{Complexl2\text{-}Space}}$ and $\Re((x|x)) \geqslant 0$ and $\Im((x|x)) = 0$ and $(x|y) = \overline{(y|x)}$ and $((x+y)|z) = (x|z) + (y|z)$ and $((a \cdot x)|y) = a \cdot (x|y)$.

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One can verify that Complexl2-Space is complex unitary space-like.

Next we state the proposition

(3) For every sequence v_1 of Complexl2-Space such that v_1 is Cauchy holds v_1 is convergent.

Let us mention that Complexl2-Space is Hilbert.

2. Some Corollaries of Complex Sequences

Next we state a number of propositions:

(4) For all Complexes z_1, z_2 such that $\Re(z_1) \cdot \Im(z_2) = \Re(z_2) \cdot \Im(z_1)$ and $\Re(z_1) \cdot \Re(z_2) + \Im(z_1) \cdot \Im(z_2) \geqslant 0$ holds $|z_1 + z_2| = |z_1| + |z_2|$.

(5) For all Complexes x, y holds $2 \cdot |x \cdot y| \leqslant |x|^{\mathbf{2}} + |y|^{\mathbf{2}}$.

(6) For all Complexes x, y holds $|x+y| \cdot |x+y| \leqslant 2 \cdot |x| \cdot |x| + 2 \cdot |y| \cdot |y|$ and $|x| \cdot |x| \leqslant 2 \cdot |x-y| \cdot |x-y| + 2 \cdot |y| \cdot |y|$.

(7) For every complex sequence s_1 holds $s_1 = \overline{\overline{s_1}}$.

(8) For every complex sequence s_1 holds $(\sum_{\alpha=0}^{\kappa} \overline{s_1}(\alpha))_{\kappa \in \mathbb{N}} = \overline{(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}}$.

(9) Let s_1 be a complex sequence and n be a natural number. Suppose that for every natural number i holds $\Re(s_1)(i) \geqslant 0$ and $\Im(s_1)(i) = 0$. Then $|(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}|(n) = (\sum_{\alpha=0}^{\kappa} |s_1|(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(10) For every complex sequence s_1 such that s_1 is summable holds $\sum \overline{s_1} = \overline{\sum s_1}$.

(11) For every complex sequence s_1 such that s_1 is absolutely summable holds $|\sum s_1| \leqslant \sum |s_1|$.

(12) Let s_1 be a complex sequence. Suppose s_1 is summable and for every natural number n holds $\Re(s_1)(n) \geqslant 0$ and $\Im(s_1)(n) = 0$. Then $|\sum s_1| = \sum |s_1|$.

(13) For every complex sequence s_1 and for every natural number n holds $\Re(s_1 \, \overline{s_1})(n) \geqslant 0$ and $\Im(s_1 \, \overline{s_1})(n) = 0$.

(14) Let s_1 be a complex sequence. Suppose s_1 is absolutely summable and $\sum |s_1| = 0$. Let n be a natural number. Then $s_1(n) = 0_{\mathbb{C}}$.

(15) For every complex sequence s_1 holds $|s_1| = |\overline{s_1}|$.

(16) Let c be a Complex and s_1 be a complex sequence. Suppose s_1 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number m holds $r_1(m) = |s_1(m) - c| \cdot |s_1(m) - c|$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c| \cdot |\lim s_1 - c|$.

(17) Let c be a Complex, s_2 be a sequence of real numbers, and s_1 be a complex sequence. Suppose s_1 is convergent and s_2 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number i

holds $r_1(i) = |s_1(i) - c| \cdot |s_1(i) - c| + s_2(i)$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c| \cdot |\lim s_1 - c| + \lim s_2$.

(18) Let c be a Complex and s_1 be a complex sequence. Suppose s_1 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number m holds $r_1(m) = |s_1(m) - c| \cdot |s_1(m) - c|$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c| \cdot |\lim s_1 - c|$.

(19) Let c be a Complex, s_2 be a sequence of real numbers, and s_1 be a complex sequence. Suppose s_1 is convergent and s_2 is convergent. Let r_1 be a sequence of real numbers. Suppose that for every natural number i holds $r_1(i) = |s_1(i) - c| \cdot |s_1(i) - c| + s_2(i)$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c| \cdot |\lim s_1 - c| + \lim s_2$.

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