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Banach Space of Absolute Summable Complex Sequences

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Summary. An extension of [16]. As the example of complex norm spaces, I introduced the arithmetic addition and multiplication in the set of absolute summable complex sequences and also introduced the norm.

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The terminology and notation used in this paper are introduced in the following articles: [18], [20], [6], [2], [17], [9], [21], [4], [5], [19], [13], [11], [10], [14], [3], [1], [12], [15], [7], and [8].

1. Complex-L1-Space: The Space of Absolute Summable Complex Sequences

The subset the set of l1-complex sequences of the linear space of complex sequences is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ the set of l1-complex sequences if and only if $x \in$ the set of complex sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is absolutely summable.

The following proposition is true

(1) Let c be a Complex, s_1 be a complex sequence, and r_1 be a sequence of real numbers. Suppose s_1 is convergent and for every natural number i holds $r_1(i) = |s_1(i) - c|$. Then r_1 is convergent and $\lim r_1 = |\lim s_1 - c|$.

Let us note that the set of l1-complex sequences is non empty.

Let us observe that the set of l1-complex sequences is linearly closed.

Next we state the proposition

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(2) ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences)⟩ is a subspace of the linear space of complex sequences.

Let us note that ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences)⟩ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

We now state the proposition

(3) ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences)⟩ is a complex linear space.

The function cl_norm from the set of l1-complex sequences into $\mathbb{R}$ is defined as follows:

(Def. 2) For every set x such that $x \in$ the set of l1-complex sequences holds $\mathrm{cl\_norm}(x) = \sum |\mathrm{id_{seq}}(x)|$.

Let X be a non empty set, let Z be an element of X, let A be a binary operation on X, let M be a function from $[: \mathbb{C}, X :]$ into X, and let N be a function from X into $\mathbb{R}$. Note that $\langle X, Z, A, M, N \rangle$ is non empty.

We now state four propositions:

(4) Let l be a complex normed space structure. Suppose ⟨the carrier of l, the zero of l, the addition of l, the external multiplication of l⟩ is a complex linear space. Then l is a complex linear space.

(5) Let c_1 be a complex sequence. Suppose that for every natural number n holds $c_1(n) = 0_{\mathbb{C}}$. Then c_1 is absolutely summable and $\sum |c_1| = 0$.

(6) Let c_1 be a complex sequence. Suppose c_1 is absolutely summable and $\sum |c_1| = 0$. Let n be a natural number. Then $c_1(n) = 0_{\mathbb{C}}$.

(7) ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences), cl_norm⟩ is a complex linear space.

The non empty complex normed space structure Complex-l1-Space is defined by the condition (Def. 3).

(Def. 3) Complex-l1-Space = ⟨the set of l1-complex sequences, Zero_(the set of l1-complex sequences, the linear space of complex sequences), Add_(the set of l1-complex sequences, the linear space of complex sequences), Mult_(the set of l1-complex sequences, the linear space of complex sequences), cl_norm⟩.

2. Complex-L1-Space is Banach

One can prove the following propositions:

(8) The carrier of Complex-l1-Space = the set of l1-complex sequences and for every set x holds x is a vector of Complex-l1-Space iff x is a complex sequence and $\mathrm{id}_{\mathrm{seq}}(x)$ is absolutely summable and $0_{\text{Complex-l1-Space}} =$ CZeroseq and for every vector u of Complex-l1-Space holds $u = \mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of Complex-l1-Space holds $u+v = \mathrm{id}_{\mathrm{seq}}(u)+\mathrm{id}_{\mathrm{seq}}(v)$ and for every Complex p and for every vector u of Complex-l1-Space holds $p \cdot u = p\ \mathrm{id}_{\mathrm{seq}}(u)$ and for every vector u of Complex-l1-Space holds $-u = -\mathrm{id}_{\mathrm{seq}}(u)$ and $\mathrm{id}_{\mathrm{seq}}(-u) = -\mathrm{id}_{\mathrm{seq}}(u)$ and for all vectors u, v of Complex-l1-Space holds $u - v = \mathrm{id}_{\mathrm{seq}}(u) - \mathrm{id}_{\mathrm{seq}}(v)$ and for every vector v of Complex-l1-Space holds $\mathrm{id}_{\mathrm{seq}}(v)$ is absolutely summable and for every vector v of Complex-l1-Space holds $\|v\| = \sum |\,\mathrm{id}_{\mathrm{seq}}(v)|$.

(9) Let x, y be points of Complex-l1-Space and p be a Complex. Then $\|x\| = 0$ iff $x = 0_{\text{Complex-l1-Space}}$ and $0 \leqslant \|x\|$ and $\|x + y\| \leqslant \|x\| + \|y\|$ and $\|p \cdot x\| = |p| \cdot \|x\|$.

Let us observe that Complex-l1-Space is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, and right complementable.

Let X be a non empty complex normed space structure and let x, y be points of X. The functor $\rho(x, y)$ yielding a real number is defined as follows:

(Def. 4) $\rho(x, y) = \|x - y\|$.

Let C_1 be a non empty complex normed space structure and let s_2 be a sequence of C_1. We say that s_2 is CCauchy if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let r_2 be a real number. Suppose $r_2 > 0$. Then there exists a natural number k_1 such that for all natural numbers n_1, m_1 if $n_1 \geqslant k_1$ and $m_1 \geqslant k_1$, then $\rho(s_2(n_1), s_2(m_1)) < r_2$.

We introduce s_1 is Cauchy sequence by norm as a synonym of s_2 is CCauchy.

In the sequel N_1 is a non empty complex normed space and s_1 is a sequence of N_1.

One can prove the following propositions:

(10) s_1 is Cauchy sequence by norm if and only if for every real number r such that $r > 0$ there exists a natural number k such that for all natural numbers n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|s_1(n) - s_1(m)\| < r$.

(11) For every sequence v_1 of Complex-l1-Space such that v_1 is Cauchy sequence by norm holds v_1 is convergent.

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