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Some Set Series in Finite Topological Spaces. Fundamental Concepts for Image Processing

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Summary. First we give a definition of "inflation" of a set in finite topological spaces. Then a concept of "deflation" of a set is also defined. In the remaining part, we give a concept of the "set series" for a subset of a finite topological space. Using this, we can define a series of neighbourhoods for each point in the space. The work is done according to [7].

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The articles [9], [5], [10], [2], [8], [1], [12], [11], [3], [4], and [6] provide the notation and terminology for this paper.

We adopt the following rules: T denotes a non empty finite topology space, A, B denote subsets of T, and x, y denote elements of T.

Let us consider T and let A be a subset of T. The functor A^d yields a subset of T and is defined by:

(Def. 1) $A^d = \{x; x$ ranges over elements of T: $\bigwedge_{y\,:\,\text{element of } T} (y \in A^{\rm c} \Rightarrow x \notin U(y))\}$.

We now state a number of propositions:

(1) If T is filled, then $A \subseteq A^f$.

(2) $x \in A^d$ iff for every y such that $y \in A^{\rm c}$ holds $x \notin U(y)$.

(3) If T is filled, then $A^d \subseteq A$.

(4) $A^d = ((A^{\rm c})^f)^{\rm c}$.

(5) If $A \subseteq B$, then $A^f \subseteq B^f$.

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(6) If $A \subseteq B$, then $A^d \subseteq B^d$.
(7) $(A \cap B)^b \subseteq A^b \cap B^b$.
(8) $(A \cup B)^b = A^b \cup B^b$.
(9) $A^i \cup B^i \subseteq (A \cup B)^i$.
(10) $A^i \cap B^i = (A \cap B)^i$.
(11) $A^f \cup B^f = (A \cup B)^f$.
(12) $A^d \cap B^d = A \cap B^d$.

Let T be a non empty finite topology space and let A be a subset of T. The functor $\mathrm{Fcl}(A)$ yields a function from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ and is defined as follows:

(Def. 2) For every natural number n and for every subset B of T such that $B = (\mathrm{Fcl}(A))(n)$ holds $(\mathrm{Fcl}(A))(n+1) = B^b$ and $(\mathrm{Fcl}(A))(0) = A$.

Let T be a non empty finite topology space, let A be a subset of T, and let n be a natural number. The functor $\mathrm{Fcl}(A, n)$ yields a subset of T and is defined by:

(Def. 3) $\mathrm{Fcl}(A, n) = (\mathrm{Fcl}(A))(n)$.

Let T be a non empty finite topology space and let A be a subset of T. The functor $\mathrm{Fint}(A)$ yields a function from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ and is defined by:

(Def. 4) For every natural number n and for every subset B of T such that $B = (\mathrm{Fint}(A))(n)$ holds $(\mathrm{Fint}(A))(n+1) = B^i$ and $(\mathrm{Fint}(A))(0) = A$.

Let T be a non empty finite topology space, let A be a subset of T, and let n be a natural number. The functor $\mathrm{Fint}(A, n)$ yields a subset of T and is defined as follows:

(Def. 5) $\mathrm{Fint}(A, n) = (\mathrm{Fint}(A))(n)$.

The following propositions are true:

(13) For every natural number n holds $\mathrm{Fcl}(A, n+1) = (\mathrm{Fcl}(A, n))^b$.
(14) $\mathrm{Fcl}(A, 0) = A$.
(15) $\mathrm{Fcl}(A, 1) = A^b$.
(16) $\mathrm{Fcl}(A, 2) = (A^b)^b$.
(17) For every natural number n holds $\mathrm{Fcl}(A \cup B, n) = \mathrm{Fcl}(A, n) \cup \mathrm{Fcl}(B, n)$.
(18) For every natural number n holds $\mathrm{Fint}(A, n+1) = (\mathrm{Fint}(A, n))^i$.
(19) $\mathrm{Fint}(A, 0) = A$.
(20) $\mathrm{Fint}(A, 1) = A^i$.
(21) $\mathrm{Fint}(A, 2) = (A^i)^i$.
(22) For every natural number n holds $\mathrm{Fint}(A \cap B, n) = \mathrm{Fint}(A, n) \cap \mathrm{Fint}(B, n)$.
(23) If T is filled, then for every natural number n holds $A \subseteq \mathrm{Fcl}(A, n)$.
(24) If T is filled, then for every natural number n holds $\mathrm{Fint}(A, n) \subseteq A$.

(25) If T is filled, then for every natural number n holds $\mathrm{Fcl}(A, n) \subseteq \mathrm{Fcl}(A, n+1)$.

(26) If T is filled, then for every natural number n holds $\mathrm{Fint}(A, n+1) \subseteq \mathrm{Fint}(A, n)$.

(27) For every natural number n holds $(\mathrm{Fint}(A^{\mathrm{c}}, n))^{\mathrm{c}} = \mathrm{Fcl}(A, n)$.

(28) For every natural number n holds $(\mathrm{Fcl}(A^{\mathrm{c}}, n))^{\mathrm{c}} = \mathrm{Fint}(A, n)$.

(29) For every natural number n holds $\mathrm{Fcl}(A, n) \cup \mathrm{Fcl}(B, n) = (\mathrm{Fint}((A \cup B)^{\mathrm{c}}, n))^{\mathrm{c}}$.

(30) For every natural number n holds $\mathrm{Fint}(A, n) \cap \mathrm{Fint}(B, n) = (\mathrm{Fcl}((A \cap B)^{\mathrm{c}}, n))^{\mathrm{c}}$.

Let T be a non empty finite topology space and let A be a subset of T. The functor $\mathrm{Finf}(A)$ yielding a function from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ is defined by:

(Def. 6) For every natural number n and for every subset B of T such that $B = (\mathrm{Finf}(A))(n)$ holds $(\mathrm{Finf}(A))(n+1) = B^f$ and $(\mathrm{Finf}(A))(0) = A$.

Let T be a non empty finite topology space, let A be a subset of T, and let n be a natural number. The functor $\mathrm{Finf}(A, n)$ yielding a subset of T is defined as follows:

(Def. 7) $\mathrm{Finf}(A, n) = (\mathrm{Finf}(A))(n)$.

Let T be a non empty finite topology space and let A be a subset of T. The functor $\mathrm{Fdfl}(A)$ yields a function from $\mathbb{N}$ into $2^{\text{the carrier of } T}$ and is defined as follows:

(Def. 8) For every natural number n and for every subset B of T such that $B = (\mathrm{Fdfl}(A))(n)$ holds $(\mathrm{Fdfl}(A))(n+1) = B^d$ and $(\mathrm{Fdfl}(A))(0) = A$.

Let T be a non empty finite topology space, let A be a subset of T, and let n be a natural number. The functor $\mathrm{Fdfl}(A, n)$ yields a subset of T and is defined as follows:

(Def. 9) $\mathrm{Fdfl}(A, n) = (\mathrm{Fdfl}(A))(n)$.

Next we state a number of propositions:

(31) For every natural number n holds $\mathrm{Finf}(A, n+1) = (\mathrm{Finf}(A, n))^f$.

(32) $\mathrm{Finf}(A, 0) = A$.

(33) $\mathrm{Finf}(A, 1) = A^f$.

(34) $\mathrm{Finf}(A, 2) = (A^f)^f$.

(35) For every natural number n holds $\mathrm{Finf}(A \cup B, n) = \mathrm{Finf}(A, n) \cup \mathrm{Finf}(B, n)$.

(36) If T is filled, then for every natural number n holds $A \subseteq \mathrm{Finf}(A, n)$.

(37) If T is filled, then for every natural number n holds $\mathrm{Finf}(A, n) \subseteq \mathrm{Finf}(A, n+1)$.

(38) For every natural number n holds $\mathrm{Fdfl}(A, n+1) = \mathrm{Fdfl}(A, n)^d$.

(39) $\mathrm{Fdfl}(A, 0) = A$.

(40) $\mathrm{Fdfl}(A, 1) = A^d$.

(41) $\mathrm{Fdfl}(A, 2) = (A^d)^d$.

(42) For every natural number n holds $\mathrm{Fdfl}(A \cap B, n) = \mathrm{Fdfl}(A, n) \cap \mathrm{Fdfl}(B, n)$.

(43) If T is filled, then for every natural number n holds $\mathrm{Fdfl}(A, n) \subseteq A$.

(44) If T is filled, then for every natural number n holds $\mathrm{Fdfl}(A, n+1) \subseteq \mathrm{Fdfl}(A, n)$.

(45) For every natural number n holds $\mathrm{Fdfl}(A, n) = (\mathrm{Finf}(A^{\mathrm{c}}, n))^{\mathrm{c}}$.

(46) For every natural number n holds $\mathrm{Fdfl}(A, n) \cap \mathrm{Fdfl}(B, n) = (\mathrm{Finf}((A \cap B)^{\mathrm{c}}, n))^{\mathrm{c}}$.

Let T be a non empty finite topology space, let n be a natural number, and let x be an element of T. The functor $U(x, n)$ yields a subset of T and is defined as follows:

(Def. 10) $U(x, n) = \mathrm{Finf}(U(x), n)$.

Next we state two propositions:

(47) $U(x, 0) = U(x)$.

(48) For every natural number n holds $U(x, n+1) = (U(x, n))^f$.

Let S, T be non empty finite topology spaces. We say that S, T are mutually symmetric if and only if the conditions (Def. 11) are satisfied.

(Def. 11)(i) The carrier of $S =$ the carrier of T, and

(ii) for all sets x, y such that $x \in$ the carrier of S and $y \in$ the carrier of T holds $y \in$ (the neighbour-map of S)(x) iff $x \in$ (the neighbour-map of T)(y).

Let us note that the predicate S, T are mutually symmetric is symmetric.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[6] Hiroshi Imura and Masayoshi Eguchi. Finite topological spaces. *Formalized Mathematics*, 3(**2**):189–193, 1992.

[7] Yatsuka Nakamura. Finite topology concept for discrete spaces. In H. Umegaki, editor, *Proceedings of the Eleventh Symposium on Applied Functional Analysis*, pages 111–116, Noda-City, Chiba, Japan, 1988. Science University of Tokyo.

[8] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[10] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[11] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[12] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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