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The Exponential Function on Banach Algebra

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Summary. In this article, the basic properties of the exponential function on Banach algebra are described.

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The notation and terminology used here are introduced in the following papers: [17], [19], [20], [3], [4], [2], [16], [5], [1], [18], [9], [11], [12], [8], [6], [7], [13], [10], [21], [14], and [15].

For simplicity, we use the following convention: X denotes a Banach algebra, p denotes a real number, w, z, z_1, z_2 denote elements of X, k, l, m, n denote natural numbers, s_1, s_2, s_3, s, s' denote sequences of X, and r_1 denotes a sequence of real numbers.

Let X be a non empty normed algebra structure and let x, y be elements of X. We say that x, y are commutative if and only if:

(Def. 1) $x \cdot y = y \cdot x$.

Let us note that the predicate x, y are commutative is symmetric.

Next we state a number of propositions:

(1) If s_2 is convergent and s_3 is convergent and $\lim(s_2 - s_3) = 0_X$, then $\lim s_2 = \lim s_3$.

(2) For every z such that for every natural number n holds $s(n) = z$ holds $\lim s = z$.

(3) If s is convergent and s' is convergent, then $s \cdot s'$ is convergent.

(4) If s is convergent, then $z \cdot s$ is convergent.

(5) If s is convergent, then $s \cdot z$ is convergent.

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(6) If s is convergent, then $\lim(z \cdot s) = z \cdot \lim s$.

(7) If s is convergent, then $\lim(s \cdot z) = \lim s \cdot z$.

(8) If s is convergent and s' is convergent, then $\lim(s \cdot s') = \lim s \cdot \lim s'$.

(9) $(\sum_{\alpha=0}^{\kappa}(z \cdot s_1)(\alpha))_{\kappa \in \mathbb{N}} = z \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}$ and $(\sum_{\alpha=0}^{\kappa}(s_1 \cdot z)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}} \cdot z$.

(10) $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa \in \mathbb{N}}(k)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa \in \mathbb{N}}(k)$.

(11) If for every n such that $n \leqslant m$ holds $s_2(n) = s_3(n)$, then $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa \in \mathbb{N}}(m) = (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa \in \mathbb{N}}(m)$.

(12) If for every n holds $\|s_1(n)\| \leqslant r_1(n)$ and r_1 is convergent and $\lim r_1 = 0$, then s_1 is convergent and $\lim s_1 = 0_X$.

Let us consider X and let z be an element of X. The functor $z\,\mathrm{ExpSeq}$ yielding a sequence of X is defined as follows:

(Def. 2) For every n holds $z\,\mathrm{ExpSeq}(n) = \frac{1}{n!} \cdot z_{\mathbb{N}}^{n}$.

The scheme *ExNormSpace CASE* deals with a non empty Banach algebra $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

For every k there exists a sequence s_1 of $\mathcal{A}$ such that for every n holds if $n \leqslant k$, then $s_1(n) = \mathcal{F}(k, n)$ and if $n > k$, then $s_1(n) = 0_{\mathcal{A}}$

for all values of the parameters.

Next we state the proposition

(13) For every k such that $0 < k$ holds $(k -' 1)! \cdot k = k!$ and for all m, k such that $k \leqslant m$ holds $(m -' k)! \cdot ((m + 1) - k) = ((m + 1) -' k)!$.

Let n be a natural number. The functor $\mathrm{Coef}\,n$ yields a sequence of real numbers and is defined by:

(Def. 3) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Coef}\,n)(k) = \frac{n!}{k! \cdot (n -' k)!}$ and if $k > n$, then $(\mathrm{Coef}\,n)(k) = 0$.

Let n be a natural number. The functor $\mathrm{Coef\_e}\,n$ yielding a sequence of real numbers is defined by:

(Def. 4) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Coef\_e}\,n)(k) = \frac{1}{k! \cdot (n -' k)!}$ and if $k > n$, then $(\mathrm{Coef\_e}\,n)(k) = 0$.

Let us consider X, s_1. The functor $\mathrm{Shift}\,s_1$ yielding a sequence of X is defined as follows:

(Def. 5) $(\mathrm{Shift}\,s_1)(0) = 0_X$ and for every natural number k holds $(\mathrm{Shift}\,s_1)(k + 1) = s_1(k)$.

Let us consider n, let us consider X, and let z, w be elements of X. The functor $\mathrm{Expan}(n, z, w)$ yields a sequence of X and is defined by:

(Def. 6) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan}(n, z, w))(k) = (\mathrm{Coef}\,n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{n -' k}$ and if $n < k$, then $(\mathrm{Expan}(n, z, w))(k) = 0_X$.

Let us consider n, let us consider X, and let z, w be elements of X. The functor $\mathrm{Expan\_e}(n, z, w)$ yields a sequence of X and is defined as follows:

(Def. 7) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Expan\_e}(n,z,w))(k) = (\mathrm{Coef\_e}\, n)(k) \cdot z_{\mathbb{N}}^{k} \cdot w_{\mathbb{N}}^{n -' k}$ and if $n < k$, then $(\mathrm{Expan\_e}(n,z,w))(k) = 0_X$.

Let us consider n, let us consider X, and let z, w be elements of X. The functor $\mathrm{Alfa}(n,z,w)$ yields a sequence of X and is defined as follows:

(Def. 8) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Alfa}(n,z,w))(k) = z\,\mathrm{ExpSeq}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n -' k)$ and if $n < k$, then $(\mathrm{Alfa}(n,z,w))(k) = 0_X$.

Let us consider X, let z, w be elements of X, and let n be a natural number. The functor $\mathrm{Conj}(n,z,w)$ yields a sequence of X and is defined by:

(Def. 9) For every natural number k holds if $k \leqslant n$, then $(\mathrm{Conj}(n,z,w))(k) = z\,\mathrm{ExpSeq}(k) \cdot ((\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n -' k))$ and if $n < k$, then $(\mathrm{Conj}(n,z,w))(k) = 0_X$.

One can prove the following propositions:

(14) $z\,\mathrm{ExpSeq}(n+1) = \frac{1}{n+1} \cdot z \cdot z\,\mathrm{ExpSeq}(n)$ and $z\,\mathrm{ExpSeq}(0) = \mathbf{1}_X$ and $\|z\,\mathrm{ExpSeq}(n)\| \leqslant \|z\|\,\mathrm{ExpSeq}(n)$.

(15) If $0 < k$, then $(\mathrm{Shift}\, s_1)(k) = s_1(k -' 1)$.

(16) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Shift}\, s_1)(\alpha))_{\kappa\in\mathbb{N}}(k) + s_1(k)$.

(17) For all z, w such that z, w are commutative holds $(z+w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan}(n,z,w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(18) $\mathrm{Expan\_e}(n,z,w) = \frac{1}{n!} \cdot \mathrm{Expan}(n,z,w)$.

(19) For all z, w such that z, w are commutative holds $\frac{1}{n!} \cdot (z+w)_{\mathbb{N}}^{n} = (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan\_e}(n,z,w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(20) $0_X\,\mathrm{ExpSeq}$ is norm-summable and $\sum(0_X\,\mathrm{ExpSeq}) = \mathbf{1}_X$.

Let us consider X and let z be an element of X. Observe that $z\,\mathrm{ExpSeq}$ is norm-summable.

Next we state a number of propositions:

(21) $z\,\mathrm{ExpSeq}(0) = \mathbf{1}_X$ and $(\mathrm{Expan}(0,z,w))(0) = \mathbf{1}_X$.

(22) If $l \leqslant k$, then $(\mathrm{Alfa}(k+1,z,w))(l) = (\mathrm{Alfa}(k,z,w))(l) + (\mathrm{Expan\_e}(k+1,z,w))(l)$.

(23) $(\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(k+1,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k) + (\sum_{\alpha=0}^{\kappa}(\mathrm{Expan\_e}(k+1,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(24) $z\,\mathrm{ExpSeq}(k) = (\mathrm{Expan\_e}(k,z,w))(k)$.

(25) For all z, w such that z, w are commutative holds $(\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Alfa}(n,z,w))(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(26) For all z, w such that z, w are commutative holds $(\sum_{\alpha=0}^{\kappa} z\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) \cdot (\sum_{\alpha=0}^{\kappa} w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) - (\sum_{\alpha=0}^{\kappa} z + w\,\mathrm{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Conj}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(27) $0 \leqslant \|z\|\,\mathrm{ExpSeq}(n)$.

(28) $\|(\sum_{\alpha=0}^{\kappa} z \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)$ and $(\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k) \leqslant \sum(\|z\| \operatorname{ExpSeq})$ and $\|(\sum_{\alpha=0}^{\kappa} z \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(k)\| \leqslant \sum(\|z\| \operatorname{ExpSeq})$.

(29) $1 \leqslant \sum(\|z\| \operatorname{ExpSeq})$.

(30) $|(\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$ and if $n \leqslant m$, then $|(\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m)-(\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|z\| \operatorname{ExpSeq}(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(31) $|(\sum_{\alpha=0}^{\kappa}\|\operatorname{Conj}(k,z,w)\|(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}\|\operatorname{Conj}(k,z,w)\|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(32) For every real number p such that $p > 0$ there exists n such that for every k such that $n \leqslant k$ holds $|(\sum_{\alpha=0}^{\kappa}\|\operatorname{Conj}(k,z,w)\|(\alpha))_{\kappa\in\mathbb{N}}(k)| < p$.

(33) For every s_1 such that for every k holds $s_1(k) = (\sum_{\alpha=0}^{\kappa}(\operatorname{Conj}(k,z,w))(\alpha))_{\kappa\in\mathbb{N}}(k)$ holds s_1 is convergent and $\lim s_1 = 0_X$.

Let X be a Banach algebra. The functor $\exp X$ yielding a function from the carrier of X into the carrier of X is defined by:

(Def. 10) For every element z of the carrier of X holds $(\exp X)(z) = \sum(z \operatorname{ExpSeq})$.

Let us consider X, z. The functor $\exp z$ yields an element of X and is defined by:

(Def. 11) $\exp z = (\exp X)(z)$.

One can prove the following propositions:

(34) For every z holds $\exp z = \sum(z \operatorname{ExpSeq})$.

(35) Let given z_1, z_2. Suppose z_1, z_2 are commutative. Then $\exp(z_1 + z_2) = \exp z_1 \cdot \exp z_2$ and $\exp(z_2+z_1) = \exp z_2 \cdot \exp z_1$ and $\exp(z_1+z_2) = \exp(z_2+z_1)$ and $\exp z_1$, $\exp z_2$ are commutative.

(36) For all z_1, z_2 such that z_1, z_2 are commutative holds $z_1 \cdot \exp z_2 = \exp z_2 \cdot z_1$.

(37) $\exp(0_X) = \mathbf{1}_X$.

(38) $\exp z \cdot \exp(-z) = \mathbf{1}_X$ and $\exp(-z) \cdot \exp z = \mathbf{1}_X$.

(39) $\exp z$ is invertible and $(\exp z)^{-1} = \exp(-z)$ and $\exp(-z)$ is invertible and $(\exp(-z))^{-1} = \exp z$.

(40) For every z and for all real numbers s, t holds $s \cdot z$, $t \cdot z$ are commutative.

(41) Let given z and s, t be real numbers. Then $\exp(s \cdot z) \cdot \exp(t \cdot z) = \exp((s+t) \cdot z)$ and $\exp(t \cdot z) \cdot \exp(s \cdot z) = \exp((t+s) \cdot z)$ and $\exp((s+t) \cdot z) = \exp((t+s) \cdot z)$ and $\exp(s \cdot z)$, $\exp(t \cdot z)$ are commutative.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[6] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[7] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[8] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[10] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[11] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[12] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[13] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(**4**):449–452, 1991.
[14] Yasunari Shidama. The Banach algebra of bounded linear operators. *Formalized Mathematics*, 12(**2**):103–108, 2004.
[15] Yasunari Shidama. The series on Banach algebra. *Formalized Mathematics*, 12(**2**):131–138, 2004.
[16] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[18] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[21] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

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