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Recursive Definitions. Part II[1]

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The papers [7], [4], [9], [8], [5], [6], [1], [10], [2], [11], and [3] provide the terminology and notation for this paper.

In this paper a, b, c, d, e, z, A, B, C, D, E are sets.

Let x be a set. Let us assume that there exist sets x_1, x_2, x_3 such that $x = \langle x_1, x_2, x_3 \rangle$. The functor $x_{\mathbf{1},3}$ is defined as follows:

(Def. 1) For all sets y_1, y_2, y_3 such that $x = \langle y_1, y_2, y_3 \rangle$ holds $x_{\mathbf{1},3} = y_1$.

The functor $x_{\mathbf{2},3}$ is defined by:

(Def. 2) For all sets y_1, y_2, y_3 such that $x = \langle y_1, y_2, y_3 \rangle$ holds $x_{\mathbf{2},3} = y_2$.

The functor $x_{\mathbf{3},3}$ is defined by:

(Def. 3) For all sets y_1, y_2, y_3 such that $x = \langle y_1, y_2, y_3 \rangle$ holds $x_{\mathbf{3},3} = y_3$.

The following propositions are true:

(1) If there exist a, b, c such that $z = \langle a, b, c \rangle$, then $z = \langle z_{\mathbf{1},3}, z_{\mathbf{2},3}, z_{\mathbf{3},3} \rangle$.

(2) If $z \in [: A, B, C :]$, then $z_{\mathbf{1},3} \in A$ and $z_{\mathbf{2},3} \in B$ and $z_{\mathbf{3},3} \in C$.

(3) If $z \in [: A, B, C :]$, then $z = \langle z_{\mathbf{1},3}, z_{\mathbf{2},3}, z_{\mathbf{3},3} \rangle$.

Let x be a set. Let us assume that there exist sets x_1, x_2, x_3, x_4 such that $x = \langle x_1, x_2, x_3, x_4 \rangle$. The functor $x_{\mathbf{1},4}$ is defined by:

(Def. 4) For all sets y_1, y_2, y_3, y_4 such that $x = \langle y_1, y_2, y_3, y_4 \rangle$ holds $x_{\mathbf{1},4} = y_1$.

The functor $x_{\mathbf{2},4}$ is defined by:

(Def. 5) For all sets y_1, y_2, y_3, y_4 such that $x = \langle y_1, y_2, y_3, y_4 \rangle$ holds $x_{\mathbf{2},4} = y_2$.

The functor $x_{\mathbf{3},4}$ is defined as follows:

(Def. 6) For all sets y_1, y_2, y_3, y_4 such that $x = \langle y_1, y_2, y_3, y_4 \rangle$ holds $x_{\mathbf{3},4} = y_3$.

The functor $x_{\mathbf{4},4}$ is defined as follows:

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(Def. 7) For all sets y_1, y_2, y_3, y_4 such that $x = \langle y_1, y_2, y_3, y_4 \rangle$ holds $x_{\mathbf{4},4} = y_4$.

Next we state three propositions:

(4) If there exist a, b, c, d such that $z = \langle a, b, c, d \rangle$, then $z = \langle z_{\mathbf{1},4}, z_{\mathbf{2},4}, z_{\mathbf{3},4}, z_{\mathbf{4},4} \rangle$.

(5) If $z \in [\!:A, B, C, D:\!]$, then $z_{\mathbf{1},4} \in A$ and $z_{\mathbf{2},4} \in B$ and $z_{\mathbf{3},4} \in C$ and $z_{\mathbf{4},4} \in D$.

(6) If $z \in [\!:A, B, C, D:\!]$, then $z = \langle z_{\mathbf{1},4}, z_{\mathbf{2},4}, z_{\mathbf{3},4}, z_{\mathbf{4},4} \rangle$.

Let x be a set. Let us assume that there exist sets x_1, x_2, x_3, x_4, x_5 such that $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$. The functor $x_{\mathbf{1},5}$ is defined by:

(Def. 8) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{1},5} = y_1$.

The functor $x_{\mathbf{2},5}$ is defined by:

(Def. 9) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{2},5} = y_2$.

The functor $x_{\mathbf{3},5}$ is defined as follows:

(Def. 10) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{3},5} = y_3$.

The functor $x_{\mathbf{4},5}$ is defined as follows:

(Def. 11) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{4},5} = y_4$.

The functor $x_{\mathbf{5},5}$ is defined by:

(Def. 12) For all sets y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $x_{\mathbf{5},5} = y_5$.

The following propositions are true:

(7) If there exist a, b, c, d, e such that $z = \langle a, b, c, d, e \rangle$, then $z = \langle z_{\mathbf{1},5}, z_{\mathbf{2},5}, z_{\mathbf{3},5}, z_{\mathbf{4},5}, z_{\mathbf{5},5} \rangle$.

(8) If $z \in [\!:A, B, C, D, E:\!]$, then $z_{\mathbf{1},5} \in A$ and $z_{\mathbf{2},5} \in B$ and $z_{\mathbf{3},5} \in C$ and $z_{\mathbf{4},5} \in D$ and $z_{\mathbf{5},5} \in E$.

(9) If $z \in [\!:A, B, C, D, E:\!]$, then $z = \langle z_{\mathbf{1},5}, z_{\mathbf{2},5}, z_{\mathbf{3},5}, z_{\mathbf{4},5}, z_{\mathbf{5},5} \rangle$.

In this article we present several logical schemes. The scheme *ExFunc3Cond* deals with a set $\mathcal{A}$, three unary functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ yielding sets, and three unary predicates $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, and states that:

> There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and for every set c such that $c \in \mathcal{A}$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ and if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ and if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$

provided the parameters meet the following conditions:

- For every set c such that $c \in \mathcal{A}$ holds if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$ and if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$ and if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$, and
- For every set c such that $c \in \mathcal{A}$ holds $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$.

The scheme *ExFunc4Cond* deals with a set $\mathcal{A}$, four unary functors $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ yielding sets, and four unary predicates $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, and states that:

There exists a function f such that
(i) $\operatorname{dom} f = \mathcal{A}$, and
(ii) for every set c such that $c \in \mathcal{A}$ holds if $\mathcal{P}[c]$, then $f(c) = \mathcal{F}(c)$ and if $\mathcal{Q}[c]$, then $f(c) = \mathcal{G}(c)$ and if $\mathcal{R}[c]$, then $f(c) = \mathcal{H}(c)$ and if $\mathcal{S}[c]$, then $f(c) = \mathcal{I}(c)$

provided the following conditions are satisfied:

- Let c be a set such that $c \in \mathcal{A}$. Then
  (i) if $\mathcal{P}[c]$, then not $\mathcal{Q}[c]$,
  (ii) if $\mathcal{P}[c]$, then not $\mathcal{R}[c]$,
  (iii) if $\mathcal{P}[c]$, then not $\mathcal{S}[c]$,
  (iv) if $\mathcal{Q}[c]$, then not $\mathcal{R}[c]$,
  (v) if $\mathcal{Q}[c]$, then not $\mathcal{S}[c]$, and
  (vi) if $\mathcal{R}[c]$, then not $\mathcal{S}[c]$,
  and
- For every set c such that $c \in \mathcal{A}$ holds $\mathcal{P}[c]$ or $\mathcal{Q}[c]$ or $\mathcal{R}[c]$ or $\mathcal{S}[c]$.

The scheme *DoubleChoiceRec* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{A}$, an element $\mathcal{D}$ of $\mathcal{B}$, and a 5-ary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ and there exists a function g from $\mathbb{N}$ into $\mathcal{B}$ such that $f(0) = \mathcal{C}$ and $g(0) = \mathcal{D}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), g(n), f(n+1), g(n+1)]$

provided the parameters satisfy the following condition:

- Let n be an element of $\mathbb{N}$, x be an element of $\mathcal{A}$, and y be an element of $\mathcal{B}$. Then there exists an element x_1 of $\mathcal{A}$ and there exists an element y_1 of $\mathcal{B}$ such that $\mathcal{P}[n, x, y, x_1, y_1]$.

The scheme *LambdaRec2Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a ternary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and $f(1) = \mathcal{B}$ and for every natural number n holds $f(n+2) = \mathcal{F}(n, f(n), f(n+1))$

for all values of the parameters.

The scheme *LambdaRec2ExD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, and a ternary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and $f(1) = \mathcal{C}$ and for every natural number n holds $f(n+2) = \mathcal{F}(n, f(n), f(n+1))$

for all values of the parameters.

The scheme *LambdaRec2Un* deals with sets $\mathcal{A}$, $\mathcal{B}$, functions $\mathcal{C}$, $\mathcal{D}$, and a ternary functor $\mathcal{C}$ yielding a set, and states that:

$\mathcal{C} = \mathcal{D}$

provided the parameters meet the following requirements:

- $\operatorname{dom} \mathcal{C} = \mathbb{N}$,
- $\mathcal{C}(0) = \mathcal{A}$ and $\mathcal{C}(1) = \mathcal{B}$,
- For every natural number n holds $\mathcal{C}(n+2) = \mathcal{C}(n, \mathcal{C}(n), \mathcal{C}(n+1))$,
- $\operatorname{dom} \mathcal{D} = \mathbb{N}$,
- $\mathcal{D}(0) = \mathcal{A}$ and $\mathcal{D}(1) = \mathcal{B}$, and
- For every natural number n holds $\mathcal{D}(n+2) = \mathcal{C}(n, \mathcal{D}(n), \mathcal{D}(n+1))$.

The scheme *LambdaRec2UnD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, functions $\mathcal{D}$, $\mathcal{E}$ from $\mathbb{N}$ into $\mathcal{A}$, and a ternary functor $\mathcal{D}$ yielding an element of $\mathcal{A}$, and states that:

$\mathcal{D} = \mathcal{E}$

provided the following requirements are met:

- $\mathcal{D}(0) = \mathcal{B}$ and $\mathcal{D}(1) = \mathcal{C}$,
- For every natural number n holds $\mathcal{D}(n+2) = \mathcal{D}(n, \mathcal{D}(n), \mathcal{D}(n+1))$,
- $\mathcal{E}(0) = \mathcal{B}$ and $\mathcal{E}(1) = \mathcal{C}$, and
- For every natural number n holds $\mathcal{E}(n+2) = \mathcal{D}(n, \mathcal{E}(n), \mathcal{E}(n+1))$.

The scheme *LambdaRec3Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a 4-ary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and $f(1) = \mathcal{B}$ and $f(2) = \mathcal{C}$ and for every natural number n holds $f(n+3) = \mathcal{F}(n, f(n), f(n+1), f(n+2))$

for all values of the parameters.

The scheme *LambdaRec3ExD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ of $\mathcal{A}$, and a 4-ary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and $f(1) = \mathcal{C}$ and $f(2) = \mathcal{D}$ and for every natural number n holds $f(n+3) = \mathcal{F}(n, f(n), f(n+1), f(n+2))$

for all values of the parameters.

The scheme *LambdaRec3Un* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, functions $\mathcal{D}$, $\mathcal{E}$, and a 4-ary functor $\mathcal{D}$ yielding a set, and states that:

$\mathcal{D} = \mathcal{E}$

provided the parameters meet the following requirements:

- $\operatorname{dom} \mathcal{D} = \mathbb{N}$,
- $\mathcal{D}(0) = \mathcal{A}$ and $\mathcal{D}(1) = \mathcal{B}$ and $\mathcal{D}(2) = \mathcal{C}$,
- For every natural number n holds $\mathcal{D}(n+3) = \mathcal{D}(n, \mathcal{D}(n), \mathcal{D}(n+1), \mathcal{D}(n+2))$,
- $\operatorname{dom} \mathcal{E} = \mathbb{N}$,
- $\mathcal{E}(0) = \mathcal{A}$ and $\mathcal{E}(1) = \mathcal{B}$ and $\mathcal{E}(2) = \mathcal{C}$, and
- For every natural number n holds $\mathcal{E}(n+3) = \mathcal{D}(n, \mathcal{E}(n), \mathcal{E}(n+1), \mathcal{E}(n+2))$.

The scheme *LambdaRec3UnD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ of $\mathcal{A}$, functions $\mathcal{E}$, $\mathcal{F}$ from $\mathbb{N}$ into $\mathcal{A}$, and a 4-ary functor $\mathcal{E}$ yielding an element of $\mathcal{A}$, and states that:

$\mathcal{E} = \mathcal{F}$

provided the parameters meet the following requirements:

- $\mathcal{E}(0) = \mathcal{B}$ and $\mathcal{E}(1) = \mathcal{C}$ and $\mathcal{E}(2) = \mathcal{D}$,
- For every natural number n holds $\mathcal{E}(n+3) = \mathcal{E}(n, \mathcal{E}(n), \mathcal{E}(n+1), \mathcal{E}(n+2))$,
- $\mathcal{F}(0) = \mathcal{B}$ and $\mathcal{F}(1) = \mathcal{C}$ and $\mathcal{F}(2) = \mathcal{D}$, and
- For every natural number n holds $\mathcal{F}(n+3) = \mathcal{E}(n, \mathcal{F}(n), \mathcal{F}(n+1), \mathcal{F}(n+2))$.

The scheme *LambdaRec4Ex* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and a 5-ary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a function f such that $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and $f(1) = \mathcal{B}$ and $f(2) = \mathcal{C}$ and $f(3) = \mathcal{D}$ and for every natural number n holds $f(n+4) = \mathcal{F}(n, f(n), f(n+1), f(n+2), f(n+3))$

for all values of the parameters.

The scheme *LambdaRec4ExD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ of $\mathcal{A}$, and a 5-ary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

> There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and $f(1) = \mathcal{C}$ and $f(2) = \mathcal{D}$ and $f(3) = \mathcal{E}$ and for every natural number n holds $f(n+4) = \mathcal{F}(n, f(n), f(n+1), f(n+2), f(n+3))$

for all values of the parameters.

The scheme *LambdaRec4Un* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, functions $\mathcal{E}$, $\mathcal{F}$, and a 5-ary functor $\mathcal{E}$ yielding a set, and states that:

$\mathcal{E} = \mathcal{F}$

provided the parameters satisfy the following conditions:

- $\operatorname{dom} \mathcal{E} = \mathbb{N}$,
- $\mathcal{E}(0) = \mathcal{A}$ and $\mathcal{E}(1) = \mathcal{B}$ and $\mathcal{E}(2) = \mathcal{C}$ and $\mathcal{E}(3) = \mathcal{D}$,
- For every natural number n holds $\mathcal{E}(n+4) = \mathcal{E}(n, \mathcal{E}(n), \mathcal{E}(n+1), \mathcal{E}(n+2), \mathcal{E}(n+3))$,
- $\operatorname{dom} \mathcal{F} = \mathbb{N}$,
- $\mathcal{F}(0) = \mathcal{A}$ and $\mathcal{F}(1) = \mathcal{B}$ and $\mathcal{F}(2) = \mathcal{C}$ and $\mathcal{F}(3) = \mathcal{D}$, and
- For every natural number n holds $\mathcal{F}(n+4) = \mathcal{E}(n, \mathcal{F}(n), \mathcal{F}(n+1), \mathcal{F}(n+2), \mathcal{F}(n+3))$.

The scheme *LambdaRec4UnD* deals with a non empty set $\mathcal{A}$, elements $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ of $\mathcal{A}$, functions $\mathcal{F}$, $\mathcal{G}$ from $\mathbb{N}$ into $\mathcal{A}$, and a 5-ary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

$\mathcal{F} = \mathcal{G}$

provided the parameters meet the following requirements:

- $\mathcal{F}(0) = \mathcal{B}$ and $\mathcal{F}(1) = \mathcal{C}$ and $\mathcal{F}(2) = \mathcal{D}$ and $\mathcal{F}(3) = \mathcal{E}$,
- For every natural number n holds $\mathcal{F}(n+4) = \mathcal{F}(n, \mathcal{F}(n), \mathcal{F}(n+1), \mathcal{F}(n+2), \mathcal{F}(n+3))$,
- $\mathcal{G}(0) = \mathcal{B}$ and $\mathcal{G}(1) = \mathcal{C}$ and $\mathcal{G}(2) = \mathcal{D}$ and $\mathcal{G}(3) = \mathcal{E}$, and

- For every natural number n holds $\mathcal{G}(n+4) = \mathcal{F}(n, \mathcal{G}(n), \mathcal{G}(n+1), \mathcal{G}(n+2), \mathcal{G}(n+3))$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[5] Michał Muzalewski and Wojciech Skaba. N-tuples and Cartesian products for n=5. *Formalized Mathematics*, 2(**2**):255–258, 1991.
[6] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[8] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[9] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[10] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[11] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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