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The Taylor Expansions

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Summary. In this article, some classic theorems of calculus are described. The Taylor expansions and the logarithmic differentiation, etc. are included here.

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The terminology and notation used in this paper have been introduced in the following articles: [22], [24], [25], [4], [6], [9], [5], [11], [20], [18], [3], [8], [2], [21], [7], [1], [23], [14], [12], [10], [17], [19], [13], [15], [16], and [26].

1. The Logarithmic Differentiation Method

For simplicity, we use the following convention: n denotes a natural number, i denotes an integer, p, x, x_0, y denote real numbers, q denotes a rational number, and f denotes a partial function from $\mathbb{R}$ to $\mathbb{R}$.

Let q be an integer. The functor $\frac{q}{\mathbb{Z}}$ yields a function from $\mathbb{R}$ into $\mathbb{R}$ and is defined as follows:

(Def. 1) For every real number x holds $(\frac{q}{\mathbb{Z}})(x) = x^{q}_{\mathbb{Z}}$.

Next we state a number of propositions:

(1) For all natural numbers m, n holds $x^{n+m}_{\mathbb{Z}} = (x^{n}_{\mathbb{Z}}) \cdot x^{m}_{\mathbb{Z}}$.

(2) $\frac{n}{\mathbb{Z}}$ is differentiable in x and $(\frac{n}{\mathbb{Z}})'(x) = n \cdot x^{n-1}_{\mathbb{Z}}$.

(3) If f is differentiable in x_0, then $(\frac{n}{\mathbb{Z}}) \cdot f$ is differentiable in x_0 and $((\frac{n}{\mathbb{Z}}) \cdot f)'(x_0) = n \cdot f(x_0)^{n-1}_{\mathbb{Z}} \cdot f'(x_0)$.

(4) $\exp(-x) = \frac{1}{\exp x}$.

(5) $(\exp x)^{\frac{1}{i}}_{\mathbb{R}} = \exp(\frac{x}{i})$.

(6) For all integers m, n holds $(\exp x)^{\frac{m}{n}}_{\mathbb{R}} = \exp(\frac{m}{n} \cdot x)$.

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(7) $(\exp x)^{q}_{\mathbb{Q}} = \exp(q \cdot x)$.

(8) $(\exp x)^{p}_{\mathbb{R}} = \exp(p \cdot x)$.

(9) $(\exp 1)^{x}_{\mathbb{R}} = \exp x$ and $(\exp 1)^{x} = \exp x$ and $e^{x} = \exp x$ and $e^{x}_{\mathbb{R}} = \exp x$.

(10) $\exp(1)^{x}_{\mathbb{R}} = \exp(x)$ and $\exp(1)^{x} = \exp(x)$ and $e^{x} = \exp(x)$ and $e^{x}_{\mathbb{R}} = \exp(x)$.

(11) $e \geqslant 2$.

(12) $\log_{e} \exp x = x$.

(13) $\log_{e} \exp(x) = x$.

(14) If $y > 0$, then $\exp \log_{e} y = y$.

(15) If $y > 0$, then $\exp(\log_{e} y) = y$.

(16) exp is one-to-one and exp is differentiable on $\mathbb{R}$ and exp is differentiable on $\Omega_{\mathbb{R}}$ and for every real number x holds $\exp'(x) = \exp(x)$ and for every real number x holds $0 < \exp'(x)$ and $\operatorname{dom}\exp = \mathbb{R}$ and $\operatorname{dom}\exp = \Omega_{\mathbb{R}}$ and $\operatorname{rng}\exp =]0, +\infty[$.

Let us note that exp is one-to-one.

We now state the proposition

(17) $\exp^{-1}$ is differentiable on $\operatorname{dom}(\exp^{-1})$ and for every real number x such that $x \in \operatorname{dom}(\exp^{-1})$ holds $(\exp^{-1})'(x) = \frac{1}{x}$.

Let us mention that $]0, +\infty[$ is non empty.

Let a be a real number. The functor $\log\_(a)$ yields a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 2) $\operatorname{dom}\log\_(a) =]0, +\infty[$ and for every element d of $]0, +\infty[$ holds $(\log\_(a))(d) = \log_{a} d$.

One can prove the following three propositions:

(18) $\log\_(e) = \exp^{-1}$ and $\log\_(e)$ is one-to-one and $\operatorname{dom}\log\_(e) =]0, +\infty[$ and $\operatorname{rng}\log\_(e) = \mathbb{R}$ and $\log\_(e)$ is differentiable on $]0, +\infty[$ and for every real number x such that $x > 0$ holds $\log\_(e)$ is differentiable in x and for every element x of $]0, +\infty[$ holds $(\log\_(e))'(x) = \frac{1}{x}$ and for every element x of $]0, +\infty[$ holds $0 < (\log\_(e))'(x)$.

(19) If f is differentiable in x_0, then $\exp \cdot f$ is differentiable in x_0 and $(\exp \cdot f)'(x_0) = \exp(f(x_0)) \cdot f'(x_0)$.

(20) If f is differentiable in x_0 and $f(x_0) > 0$, then $\log\_(e) \cdot f$ is differentiable in x_0 and $(\log\_(e) \cdot f)'(x_0) = \frac{f'(x_0)}{f(x_0)}$.

Let p be a real number. The functor $^{p}_{\mathbb{R}}$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 3) $\operatorname{dom}(^{p}_{\mathbb{R}}) =]0, +\infty[$ and for every element d of $]0, +\infty[$ holds $(^{p}_{\mathbb{R}})(d) = d^{p}_{\mathbb{R}}$.

We now state two propositions:

(21) If $x > 0$, then $^{p}_{\mathbb{R}}$ is differentiable in x and $(^{p}_{\mathbb{R}})'(x) = p \cdot x^{p-1}_{\mathbb{R}}$.

(22) If f is differentiable in x_0 and $f(x_0) > 0$, then $\binom{p}{\mathbb{R}} \cdot f$ is differentiable in x_0 and $(\binom{p}{\mathbb{R}} \cdot f)'(x_0) = p \cdot f(x_0)^{p-1}_{\mathbb{R}} \cdot f'(x_0)$.

2. The Taylor Expansions

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and let Z be a subset of $\mathbb{R}$. The functor $f'(Z)$ yields a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ and is defined by:

(Def. 4) $f'(Z)(0) = f{\restriction}Z$ and for every natural number i holds $f'(Z)(i+1) = f'(Z)(i)'_{\restriction Z}$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, let n be a natural number, and let Z be a subset of $\mathbb{R}$. We say that f is differentiable n times on Z if and only if:

(Def. 5) For every natural number i such that $i \leqslant n-1$ holds $f'(Z)(i)$ is differentiable on Z.

The following proposition is true

(23) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and n be a natural number. Suppose f is differentiable n times on Z. Let m be a natural number. If $m \leqslant n$, then f is differentiable m times on Z.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, let Z be a subset of $\mathbb{R}$, and let a, b be real numbers. The functor $\mathrm{Taylor}(f, Z, a, b)$ yields a sequence of real numbers and is defined as follows:

(Def. 6) For every natural number n holds $(\mathrm{Taylor}(f, Z, a, b))(n) = \frac{f'(Z)(n)(a) \cdot (b-a)^n}{n!}$.

The following propositions are true:

(24) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and n be a natural number. Suppose f is differentiable n times on Z. Let a, b be real numbers. If $a < b$ and $]a, b[\subseteq Z$, then $f'(Z)(n){\restriction}]a, b[= f'(]a, b[)(n)$.

(25) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and Z be a subset of $\mathbb{R}$. Suppose f is differentiable n times on Z. Let a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq Z$ and $f'(Z)(n)$ is continuous on $[a, b]$ and f is differentiable $n+1$ times on $]a, b[$. Let l be a real number and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\mathrm{dom}\, g = \mathbb{R}$ and for every real number x holds $g(x) = f(b) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f, Z, x, b))(\alpha))_{\kappa \in \mathbb{N}}(n) - \frac{l \cdot (b-x)^{n+1}}{(n+1)!}$ and $f(b) - (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f, Z, a, b))(\alpha))_{\kappa \in \mathbb{N}}(n) - \frac{l \cdot (b-a)^{n+1}}{(n+1)!} = 0$. Then

(i) g is differentiable on $]a, b[$,

(ii) $g(a) = 0$,

(iii) $g(b) = 0$,

(iv) g is continuous on $[a, b]$, and

(v) for every real number x such that $x \in]a, b[$ holds $g'(x) = -\frac{f'(]a, b[)(n+1)(x) \cdot (b-x)^n}{n!} + \frac{l \cdot (b-x)^n}{n!}$.

(26) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and b, l be real numbers. Then there exists a function g from $\mathbb{R}$ into $\mathbb{R}$ such that for every real number x holds $g(x) = f(b) - (\sum_{\alpha=0}^{\kappa}(\text{Taylor}(f, Z, x, b))(\alpha))_{\kappa \in \mathbb{N}}(n) - \frac{l \cdot (b-x)^{n+1}}{(n+1)!}$.

(27) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and Z be a subset of $\mathbb{R}$. Suppose f is differentiable n times on Z. Let a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq Z$ and $f'(Z)(n)$ is continuous on $[a, b]$ and f is differentiable $n+1$ times on $]a, b[$. Then there exists a real number c such that $c \in]a, b[$ and $f(b) = (\sum_{\alpha=0}^{\kappa}(\text{Taylor}(f, Z, a, b))(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{f'(]a,b[)(n+1)(c) \cdot (b-a)^{n+1}}{(n+1)!}$.

(28) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and Z be a subset of $\mathbb{R}$. Suppose f is differentiable n times on Z. Let a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq Z$ and $f'(Z)(n)$ is continuous on $[a, b]$ and f is differentiable $n + 1$ times on $]a, b[$. Let l be a real number and g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\text{dom}\, g = \mathbb{R}$ and for every real number x holds $g(x) = f(a) - (\sum_{\alpha=0}^{\kappa}(\text{Taylor}(f, Z, x, a))(\alpha))_{\kappa \in \mathbb{N}}(n) - \frac{l \cdot (a-x)^{n+1}}{(n+1)!}$ and $f(a) - (\sum_{\alpha=0}^{\kappa}(\text{Taylor}(f, Z, b, a))(\alpha))_{\kappa \in \mathbb{N}}(n) - \frac{l \cdot (a-b)^{n+1}}{(n+1)!} = 0$. Then

(i) g is differentiable on $]a, b[$,
(ii) $g(b) = 0$,
(iii) $g(a) = 0$,
(iv) g is continuous on $[a, b]$, and
(v) for every real number x such that $x \in]a, b[$ holds $g'(x) = -\frac{f'(]a,b[)(n+1)(x) \cdot (a-x)^n}{n!} + \frac{l \cdot (a-x)^n}{n!}$.

(29) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and Z be a subset of $\mathbb{R}$. Suppose f is differentiable n times on Z. Let a, b be real numbers. Suppose $a < b$ and $[a, b] \subseteq Z$ and $f'(Z)(n)$ is continuous on $[a, b]$ and f is differentiable $n+1$ times on $]a, b[$. Then there exists a real number c such that $c \in]a, b[$ and $f(a) = (\sum_{\alpha=0}^{\kappa}(\text{Taylor}(f, Z, b, a))(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{f'(]a,b[)(n+1)(c) \cdot (a-b)^{n+1}}{(n+1)!}$.

(30) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and Z_1 be an open subset of $\mathbb{R}$. Suppose $Z_1 \subseteq Z$. Let n be a natural number. If f is differentiable n times on Z, then $f'(Z)(n) \restriction Z_1 = f'(Z_1)(n)$.

(31) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and Z_1 be an open subset of $\mathbb{R}$. Suppose $Z_1 \subseteq Z$. Let n be a natural number. Suppose f is differentiable $n + 1$ times on Z. Then f is differentiable $n + 1$ times on Z_1.

(32) Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z be a subset of $\mathbb{R}$, and x be a real number. If $x \in Z$, then for every natural number n holds $f(x) = (\sum_{\alpha=0}^{\kappa}(\text{Taylor}(f, Z, x, x))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

(33) Let n be a natural number, f be a partial function from $\mathbb{R}$ to $\mathbb{R}$, and x_0, r be real numbers. Suppose $0 < r$ and f is differentiable $n+1$ times on $]x_0 - r, x_0 + r[$. Let x be a real number. Suppose $x \in\ ]x_0 - r, x_0 + r[$. Then there exists a real number s such that $0 < s$ and $s < 1$ and $f(x) = (\sum_{\alpha=0}^{\kappa}(\mathrm{Taylor}(f,]x_0 - r, x_0 + r[, x_0, x))(\alpha))_{\kappa \in \mathbb{N}}(n) + \frac{f'(]x_0-r,x_0+r[)(n+1)(x_0+s\cdot(x-x_0))\cdot(x-x_0)^{n+1}}{(n+1)!}$.

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