The Hall Marriage Theorem

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Summary. The Marriage Theorem, as credited to Philip Hall [7], gives the necessary and sufficient condition allowing us to select a distinct element from each of a finite collection \( \{A_i\} \) of \( n \) finite subsets. This selection, called a set of different representatives (SDR), exists if and only if the marriage condition (or Hall condition) is satisfied:

\[
\forall J \subseteq \{1,\ldots,n\} \left| \bigcup_{i \in J} A_i \right| \geq |J|.
\]

The proof which is given in this article (according to Richard Rado, 1967) is based on the lemma that for finite sequences with non-trivial elements which satisfy Hall property there exists a reduction (see Def. 5) such that Hall property again holds (see Th. 29 for details).

MML Identifier: HALLMAR1.

The notation and terminology used here are introduced in the following papers: [9], [5], [10], [11], [4], [8], [2], [6], [1], and [3].

1. Preliminaries

One can prove the following proposition

(1) For all finite sets \( X, Y \) holds \( \text{card}(X \cup Y) + \text{card}(X \cap Y) = \text{card} X + \text{card} Y. \)

In this article we present several logical schemes. The scheme Regr11 deals with a natural number \( A \) and a unary predicate \( P \), and states that:

For every natural number \( k \) such that \( 1 \leq k \leq A \) holds \( P[k] \)

¹This work has been partially supported by the CALCULEMUS grant HPRN-CT-2000-00102.
provided the parameters meet the following conditions:
- \( P[A] \) and \( A \geq 2 \), and
- For every natural number \( k \) such that \( 1 \leq k \) and \( k < A \) and \( P[k + 1] \) holds \( P[k] \).

The scheme \textit{Regr2} concerns a unary predicate \( P \), and states that:
- \( P[1] \)

provided the parameters meet the following requirements:
- There exists a natural number \( n \) such that \( n > 1 \) and \( P[n] \), and
- For every natural number \( k \) such that \( k \geq 1 \) and \( P[k + 1] \) holds \( P[k] \).

Let \( F \) be a non empty set. One can check that there exists a finite sequence of elements of \( 2^F \) which is non empty and non-empty.

We now state the proposition

(2) Let \( F \) be a non empty set, \( f \) be a non-empty finite sequence of elements of \( 2^F \), and \( i \) be a natural number. If \( i \in \text{dom} \ f \), then \( f(i) \neq \emptyset \).

Let \( F \) be a finite set, let \( A \) be a finite sequence of elements of \( 2^F \), and let \( i \) be a natural number. Note that \( A(i) \) is finite.

2. Union of Finite Sequences

Let \( F \) be a set, let \( A \) be a finite sequence of elements of \( 2^F \), and let \( J \) be a set. The functor \( \bigcup_j A \) yields a set and is defined as follows:

(Def. 1) For every set \( x \) holds \( x \in \bigcup_j A \) iff there exists a set \( j \) such that \( j \in J \) and \( j \in \text{dom} A \) and \( x \in A(j) \).

Next we state two propositions:

(3) For every set \( F \) and for every finite sequence \( A \) of elements of \( 2^F \) and for every set \( J \) holds \( \bigcup_j A \subseteq F \).

(4) Let \( F \) be a finite set, \( A \) be a finite sequence of elements of \( 2^F \), and \( J, K \) be sets. If \( J \subseteq K \), then \( \bigcup_j A \subseteq \bigcup_k A \).

Let \( F \) be a finite set, let \( A \) be a finite sequence of elements of \( 2^F \), and let \( J \) be a set. One can verify that \( \bigcup_j A \) is finite.

The following propositions are true:

(5) Let \( F \) be a finite set, \( A \) be a finite sequence of elements of \( 2^F \), and \( i \) be a natural number. If \( i \in \text{dom} A \), then \( \bigcup\{i\} A = A(i) \).

(6) Let \( F \) be a finite set, \( A \) be a finite sequence of elements of \( 2^F \), and \( i, j \) be natural numbers. If \( i \in \text{dom} A \) and \( j \in \text{dom} A \), then \( \bigcup\{i,j\} A = A(i) \cup A(j) \).

(7) Let \( J \) be a set, \( F \) be a finite set, \( A \) be a finite sequence of elements of \( 2^F \), and \( i \) be a natural number. If \( i \in J \) and \( i \in \text{dom} A \), then \( A(i) \subseteq \bigcup_j A \).
3. Cut Operation for Finite Sequences

Let $F$ be a finite set, let $A$ be a finite sequence of elements of $2^F$, let $i$ be a natural number, and let $x$ be a set. The functor $\text{Cut}(A, i, x)$ yielding a finite sequence of elements of $2^F$ is defined by the conditions (Def. 2).

(Def. 2)(i) \quad \text{dom } \text{Cut}(A, i, x) = \text{dom } A, \text{ and }

(ii) \quad \text{for every natural number } k \text{ such that } k \in \text{dom } \text{Cut}(A, i, x) \text{ holds if } i = k,
then \((\text{Cut}(A, i, x))(k) = A(k) \setminus \{x\}\) and if \(i \neq k\), then \((\text{Cut}(A, i, x))(k) = A(k)\).

The following propositions are true:

(11) \quad \text{Let } F \text{ be a finite set, } A \text{ be a finite sequence of elements of } 2^F, \text{ i be a natural number, and let } x \text{ be a set. If } i \in \text{dom } A \text{ and } x \in A(i), \text{ then } \text{card}(\text{Cut}(A, i, x))(i) = \text{card } A(i) - 1.

(12) \quad \text{Let } F \text{ be a finite set, } A \text{ be a finite sequence of elements of } 2^F, \text{ i be a natural number, and let } x, J \text{ be sets. Then } \bigcup_{J \setminus \{i\}} \text{Cut}(A, i, x) = \bigcup_{J \setminus \{i\}} A.

(13) \quad \text{Let } F \text{ be a finite set, } A \text{ be a finite sequence of elements of } 2^F, \text{ i be a natural number, and let } x, J \text{ be sets. If } i \notin J, \text{ then } \bigcup_J A = \bigcup_J \text{Cut}(A, i, x).

(14) \quad \text{Let } F \text{ be a finite set, } A \text{ be a finite sequence of elements of } 2^F, \text{ i be a natural number, and let } x, J \text{ be sets. If } i \in \text{dom } \text{Cut}(A, i, x) \text{ and } J \subseteq \text{dom } \text{Cut}(A, i, x) \text{ and } i \in J, \text{ then } \bigcup_J \text{Cut}(A, i, x) = \bigcup_{J \setminus \{i\}} A \cup (A(i) \setminus \{x\}).

4. System of Different Representatives and Hall Property

Let $F$ be a finite set, let $X$ be a finite sequence of elements of $2^F$, and let $A$ be a set. We say that $A$ is a system of different representatives of $X$ if and only if the condition (Def. 3) is satisfied.

(Def. 3) \quad \text{There exists a finite sequence } f \text{ of elements of } F \text{ such that } f = A \text{ and } \text{dom } X = \text{dom } f \text{ and for every natural number } i \text{ such that } i \in \text{dom } f \text{ holds } f(i) \in X(i) \text{ and } f \text{ is one-to-one.}
Let $F$ be a finite set and let $A$ be a finite sequence of elements of $2^F$. We say that $A$ satisfies Hall condition if and only if:

(Def. 4) For every finite set $J$ such that $J \subseteq \text{dom } A$ holds $\text{card } J \leq \text{card } \bigcup_j A$.

Next we state four propositions:

(15) Let $F$ be a finite set and $A$ be a non-empty finite sequence of elements of $2^F$. If $A$ satisfies Hall condition, then $A$ is non-empty.

(16) Let $F$ be a finite set, $A$ be a finite sequence of elements of $2^F$, and $i$ be a natural number. If $i \in \text{dom } A$ and $A$ satisfies Hall condition, then $\text{card } A(i) \geq 1$.

(17) Let $F$ be a non-empty finite set and $A$ be a non-empty finite sequence of elements of $2^F$. Suppose for every natural number $i$ such that $i \in \text{dom } A$ holds $\text{card } A(i) = 1$ and $A$ satisfies Hall condition. Then there exists a set which is a system of different representatives of $A$.

(18) Let $F$ be a finite set and $A$ be a finite sequence of elements of $2^F$ such that there exists a set which is a system of different representatives of $A$. Then $A$ satisfies Hall condition.

5. Reductions and Singlifications of Finite Sequences

Let $F$ be a set, let $A$ be a finite sequence of elements of $2^F$, and let $i$ be a natural number. A finite sequence of elements of $2^F$ is said to be a reduction of $A$ at $i$-th position if:

(Def. 5) $\text{dom } i = \text{dom } A$ and for every natural number $j$ such that $j \in \text{dom } A$ and $j \neq i$ holds $A(j) = i(j)$ and $i(i) \subseteq A(i)$.

Let $F$ be a set and let $A$ be a finite sequence of elements of $2^F$. A finite sequence of elements of $2^F$ is said to be a reduction of $A$ if:

(Def. 6) $\text{dom } i = \text{dom } A$ and for every natural number $i$ such that $i \in \text{dom } A$ holds $i(i) \subseteq A(i)$.

Let $F$ be a set, let $A$ be a finite sequence of elements of $2^F$, and let $i$ be a natural number. Let us assume that $i \in \text{dom } A$ and $A(i) \neq \emptyset$. A reduction of $A$ is called a singlification of $A$ at $i$-th position if:

(Def. 7) $i(i) = 1$.

One can prove the following propositions:

(19) Let $F$ be a finite set, $A$ be a finite sequence of elements of $2^F$, and $i$ be a natural number. Then every reduction of $A$ at $i$-th position is a reduction of $A$.

(20) Let $F$ be a finite set, $A$ be a finite sequence of elements of $2^F$, $i$ be a natural number, and $x$ be a set. If $i \in \text{dom } A$ and $x \in A(i)$, then $\text{Cut}(A, i, x)$ is a reduction of $A$ at $i$-th position.
(21) Let $F$ be a finite set, $A$ be a finite sequence of elements of $2^F$, $i$ be a natural number, and $x$ be a set. If $i \in \text{dom} A$ and $x \in A(i)$, then $\text{Cut}(A, i, x)$ is a reduction of $A$.

(22) Let $F$ be a finite set, $A$ be a finite sequence of elements of $2^F$, and $B$ be a reduction of $A$. Then every reduction of $B$ is a reduction of $A$.

(23) Let $F$ be a non-empty finite set, $A$ be a non-empty finite sequence of elements of $2^F$, $i$ be a natural number, and $B$ be a singlification of $A$ at $i$-th position. If $i \in \text{dom} A$, then $B(i) \neq \emptyset$.

(24) Let $F$ be a non-empty finite set, $A$ be a non-empty finite sequence of elements of $2^F$, $i$, $j$ be natural numbers, $B$ be a singlification of $A$ at $i$-th position, and $C$ be a singlification of $B$ at $j$-th position. Suppose $i \in \text{dom} A$ and $j \in \text{dom} A$ and $C(i) \neq \emptyset$ and $B(j) \neq \emptyset$. Then $C$ is a singlification of $A$ at $j$-th position and a singlification of $A$ at $i$-th position.

(25) Let $F$ be a finite set, $A$ be a finite sequence of elements of $2^F$, and $i$ be a natural number. Then $A$ is a reduction of $A$ at $i$-th position.

(26) For every set $F$ holds every finite sequence $A$ of elements of $2^F$ is a reduction of $A$.

Let $F$ be a non-empty set and let $A$ be a finite sequence of elements of $2^F$. Let us assume that $A$ is non-empty. A reduction of $A$ is called a singlification of $A$ if:

(Def. 8) For every natural number $i$ such that $i \in \text{dom} A$ holds $\text{it}(i) = 1$.

We now state the proposition

(27) Let $F$ be a non-empty finite set, $A$ be a non-empty non-empty finite sequence of elements of $2^F$, and $f$ be a function. Then $f$ is a singlification of $A$ if and only if the following conditions are satisfied:

(i) $\text{dom} f = \text{dom} A$, and

(ii) for every natural number $i$ such that $i \in \text{dom} A$ holds $f$ is a singlification of $A$ at $i$-th position.

Let $F$ be a non-empty finite set, let $A$ be a non-empty finite sequence of elements of $2^F$, and let $k$ be a natural number. Note that every singlification of $A$ at $k$-th position is non empty.

Let $F$ be a non-empty finite set and let $A$ be a non-empty finite sequence of elements of $2^F$. One can check that every singlification of $A$ is non empty.

6. RADO’S PROOF OF THE HALL MARRIAGE THEOREM

One can prove the following propositions:

(28) Let $F$ be a non-empty finite set, $A$ be a non-empty finite sequence of elements of $2^F$, $X$ be a set, and $B$ be a reduction of $A$. Suppose $X$ is a
system of different representatives of $B$. Then $X$ is a system of different representatives of $A$.

(29) Let $F$ be a finite set and $A$ be a finite sequence of elements of $2^F$. Suppose $A$ satisfies Hall condition. Let $i$ be a natural number. If $\text{card} A(i) \geq 2$, then there exists a set $x$ such that $x \in A(i)$ and $\text{Cut}(A, i, x)$ satisfies Hall condition.

(30) Let $F$ be a finite set, $A$ be a finite sequence of elements of $2^F$, and $i$ be a natural number. If $i \in \text{dom} A$ and $A$ satisfies Hall condition, then there exists a singification of $A$ at $i$-th position which satisfies Hall condition.

(31) Let $F$ be a non empty finite set and $A$ be a non empty finite sequence of elements of $2^F$. If $A$ satisfies Hall condition, then there exists a singification of $A$ which satisfies Hall condition.

(32) Let $F$ be a non empty finite set and $A$ be a non empty finite sequence of elements of $2^F$. Then there exists a set which is a system of different representatives of $A$ if and only if $A$ satisfies Hall condition.

References


Received May 11, 2004