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Catalan Numbers

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Summary. In this paper, we define Catalan sequence (starting from 0) and prove some of its basic properties. The Catalan numbers $(0, 1, 1, 2, 5, 14, 42, \ldots)$ arise in a number of problems in combinatorics. They can be computed e.g. using the formula

$$C_n = \frac{\genfrac{}{}{0pt}{}{2n}{n}}{n+1},$$

their recursive definition is also well known:

$$C_1 = 1, \quad C_n = \Sigma_{i=1}^{n-1} C_i C_{n-i}, \quad n \geqslant 2.$$

Among other things, the Catalan numbers describe the number of ways in which parentheses can be placed in a sequence of numbers to be multiplied, two at a time.

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The articles [2], [3], [4], [1], [5], [8], [6], and [7] provide the terminology and notation for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For every natural number n such that $n > 1$ holds $n -' 1 \leqslant 2 \cdot n -' 3$.

(2) For every natural number n such that $n \geqslant 1$ holds $n -' 1 \leqslant 2 \cdot n -' 2$.

(3) For every natural number n such that $n > 1$ holds $n < 2 \cdot n -' 1$.

(4) For every natural number n such that $n > 1$ holds $(n -' 2) + 1 = n -' 1$.

(5) For every natural number n such that $n > 1$ holds $\frac{4 \cdot n \cdot n - 2 \cdot n}{n+1} > 1$.

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(6) For every natural number n such that $n > 1$ holds $(2 \cdot n -' 2)! \cdot n \cdot (n+1) < (2 \cdot n)!$.

(7) For every natural number n holds $2 \cdot (2 - \frac{3}{n+1}) < 4$.

2. Definition of Catalan Numbers

Let n be a natural number. The functor Catalan(n) yields a real number and is defined as follows:

(Def. 1) $\mathrm{Catalan}(n) = \frac{\binom{2 \cdot n -' 2}{n -' 1}}{n}$.

The following propositions are true:

(8) For every natural number n such that $n > 1$ holds $\mathrm{Catalan}(n) = \frac{(2 \cdot n -' 2)!}{(n -' 1)! \cdot n!}$.

(9) For every natural number n such that $n > 1$ holds $\mathrm{Catalan}(n) = 4 \cdot \binom{2 \cdot n -' 3}{n -' 1} - \binom{2 \cdot n -' 1}{n -' 1}$.

(10) $\mathrm{Catalan}(0) = 0$.

(11) $\mathrm{Catalan}(1) = 1$.

(12) $\mathrm{Catalan}(2) = 1$.

(13) For every natural number n holds Catalan(n) is an integer.

(14) For every natural number k such that $k \geqslant 1$ holds $\mathrm{Catalan}(k + 1) = \frac{(2 \cdot k)!}{k! \cdot (k+1)!}$.

3. Basic Properties of Catalan Numbers

We now state several propositions:

(15) For every natural number n such that $n > 1$ holds $\mathrm{Catalan}(n) < \mathrm{Catalan}(n + 1)$.

(16) For every natural number n holds $\mathrm{Catalan}(n) \leqslant \mathrm{Catalan}(n + 1)$.

(17) For every natural number n holds $\mathrm{Catalan}(n) \geqslant 0$.

(18) For every natural number n holds Catalan(n) is a natural number.

(19) For every natural number n such that $n > 0$ holds $\mathrm{Catalan}(n + 1) = 2 \cdot (2 - \frac{3}{n+1}) \cdot \mathrm{Catalan}(n)$.

Let n be a natural number. Note that Catalan(n) is natural.

Next we state the proposition

(20) For every natural number n such that $n > 0$ holds $\mathrm{Catalan}(n) > 0$.

Let n be a non empty natural number. One can verify that Catalan(n) is non empty.

One can prove the following proposition

(21) For every natural number n such that $n > 0$ holds $\mathrm{Catalan}(n+1) < 4 \cdot \mathrm{Catalan}(n)$.

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