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Complex Valued Functions Space

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Summary. This article is an extension of [9] to complex valued functions.

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The articles [14], [5], [16], [10], [17], [3], [4], [1], [12], [11], [15], [2], [8], [13], [9], [7], and [6] provide the notation and terminology for this paper.

1. Operation of Complex Functions

We adopt the following convention: x_1, x_2, z are sets, A is a non empty set, and f, g, h are elements of $\mathbb{C}^A$.

Let us consider A. The functor $+_{\mathbb{C}^A}$ yielding a binary operation on $\mathbb{C}^A$ is defined by:

(Def. 1) For all elements f, g of $\mathbb{C}^A$ holds $+_{\mathbb{C}^A}(f, g) = (+_{\mathbb{C}})^{\circ}(f, g)$.

Let us consider A. The functor $\cdot_{\mathbb{C}^A}$ yielding a binary operation on $\mathbb{C}^A$ is defined as follows:

(Def. 2) For all elements f, g of $\mathbb{C}^A$ holds $\cdot_{\mathbb{C}^A}(f, g) = (\cdot_{\mathbb{C}})^{\circ}(f, g)$.

Let us consider A. The functor $\cdot^{\mathbb{C}}_{\mathbb{C}^A}$ yielding a function from $[\mathbb{C}, \mathbb{C}^A]$ into $\mathbb{C}^A$ is defined by:

(Def. 3) For every complex number z and for every element f of $\mathbb{C}^A$ and for every element x of A holds $\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle z, f\rangle)(x) = z \cdot f(x)$.

Let us consider A. The functor $\mathbf{0}_{\mathbb{C}^A}$ yielding an element of $\mathbb{C}^A$ is defined by:

(Def. 4) $\mathbf{0}_{\mathbb{C}^A} = A \longmapsto 0_{\mathbb{C}}$.

Let us consider A. The functor $\mathbf{1}_{\mathbb{C}^A}$ yields an element of $\mathbb{C}^A$ and is defined by:

(Def. 5) $\mathbf{1}_{\mathbb{C}^A} = A \longmapsto 1_{\mathbb{C}}$.

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One can prove the following propositions:

(1) $h = +_{\mathbb{C}^A}(f, g)$ iff for every element x of A holds $h(x) = f(x) + g(x)$.

(2) $h = \cdot_{\mathbb{C}^A}(f, g)$ iff for every element x of A holds $h(x) = f(x) \cdot g(x)$.

(3) For every element x of A holds $\mathbf{1}_{\mathbb{C}^A}(x) = 1_{\mathbb{C}}$.

(4) For every element x of A holds $\mathbf{0}_{\mathbb{C}^A}(x) = 0_{\mathbb{C}}$.

(5) $\mathbf{0}_{\mathbb{C}^A} \neq \mathbf{1}_{\mathbb{C}^A}$.

In the sequel a, b denote complex numbers.

The following proposition is true

(6) $h = \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle)$ iff for every element x of A holds $h(x) = a \cdot f(x)$.

In the sequel u, v, w are vectors of $\langle \mathbb{C}^A, \mathbf{0}_{\mathbb{C}^A}, +_{\mathbb{C}^A}, \cdot^{\mathbb{C}}_{\mathbb{C}^A}\rangle$.

One can prove the following propositions:

(7) $+_{\mathbb{C}^A}(f, g) = +_{\mathbb{C}^A}(g, f)$.

(8) $+_{\mathbb{C}^A}(f, +_{\mathbb{C}^A}(g, h)) = +_{\mathbb{C}^A}(+_{\mathbb{C}^A}(f, g), h)$.

(9) $\cdot_{\mathbb{C}^A}(f, g) = \cdot_{\mathbb{C}^A}(g, f)$.

(10) $\cdot_{\mathbb{C}^A}(f, \cdot_{\mathbb{C}^A}(g, h)) = \cdot_{\mathbb{C}^A}(\cdot_{\mathbb{C}^A}(f, g), h)$.

(11) $\cdot_{\mathbb{C}^A}(\mathbf{1}_{\mathbb{C}^A}, f) = f$.

(12) $+_{\mathbb{C}^A}(\mathbf{0}_{\mathbb{C}^A}, f) = f$.

(13) $+_{\mathbb{C}^A}(f, \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle -1_{\mathbb{C}}, f\rangle)) = \mathbf{0}_{\mathbb{C}^A}$.

(14) $\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle 1_{\mathbb{C}}, f\rangle) = f$.

(15) $\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, f\rangle)\rangle) = \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a \cdot b, f\rangle)$.

(16) $+_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, f\rangle)) = \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a + b, f\rangle)$.

(17) $\cdot_{\mathbb{C}^A}(f, +_{\mathbb{C}^A}(g, h)) = +_{\mathbb{C}^A}(\cdot_{\mathbb{C}^A}(f, g), \cdot_{\mathbb{C}^A}(f, h))$.

(18) $\cdot_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), g) = \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, \cdot_{\mathbb{C}^A}(f, g)\rangle)$.

2. Complex Linear Space of Complex Valued Functions

One can prove the following propositions:

(19) There exist f, g such that

(i) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1_{\mathbb{C}}$ and if $z \neq x_1$, then $f(z) = 0_{\mathbb{C}}$, and

(ii) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0_{\mathbb{C}}$ and if $z \neq x_1$, then $g(z) = 1_{\mathbb{C}}$.

(20) Suppose that

(i) $x_1 \in A$,

(ii) $x_2 \in A$,

(iii) $x_1 \neq x_2$,

(iv) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1_{\mathbb{C}}$ and if $z \neq x_1$, then $f(z) = 0_{\mathbb{C}}$, and

(v) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0_{\mathbb{C}}$ and if $z \neq x_1$, then $g(z) = 1_{\mathbb{C}}$.
Let given a, b. If $+_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle)) = \mathbf{0}_{\mathbb{C}^A}$, then $a = 0_{\mathbb{C}}$ and $b = 0_{\mathbb{C}}$.

(21) If $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$, then there exist f, g such that for all a, b such that $+_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle)) = \mathbf{0}_{\mathbb{C}^A}$ holds $a = 0_{\mathbb{C}}$ and $b = 0_{\mathbb{C}}$.

(22) Suppose that
(i) $A = \{x_1, x_2\}$,
(ii) $x_1 \neq x_2$,
(iii) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1_{\mathbb{C}}$ and if $z \neq x_1$, then $f(z) = 0_{\mathbb{C}}$, and
(iv) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0_{\mathbb{C}}$ and if $z \neq x_1$, then $g(z) = 1_{\mathbb{C}}$.
Let given h. Then there exist a, b such that $h = +_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle))$.

(23) If $A = \{x_1, x_2\}$ and $x_1 \neq x_2$, then there exist f, g such that for every h there exist a, b such that $h = +_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle))$.

(24) Suppose $A = \{x_1, x_2\}$ and $x_1 \neq x_2$. Then there exist f, g such that for all a, b such that $+_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle)) = \mathbf{0}_{\mathbb{C}^A}$ holds $a = 0_{\mathbb{C}}$ and $b = 0_{\mathbb{C}}$ and for every h there exist a, b such that $h = +_{\mathbb{C}^A}(\cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle a, f\rangle), \cdot^{\mathbb{C}}_{\mathbb{C}^A}(\langle b, g\rangle))$.

(25) $\langle \mathbb{C}^A, \mathbf{0}_{\mathbb{C}^A}, +_{\mathbb{C}^A}, \cdot^{\mathbb{C}}_{\mathbb{C}^A}\rangle$ is a complex linear space.

Let us consider A. The functor ComplexVectSpace(A) yields a strict complex linear space and is defined by:

(Def. 6) ComplexVectSpace(A) $= \langle \mathbb{C}^A, \mathbf{0}_{\mathbb{C}^A}, +_{\mathbb{C}^A}, \cdot^{\mathbb{C}}_{\mathbb{C}^A}\rangle$.

We now state the proposition

(26) There exists a strict complex linear space V and there exist vectors u, v of V such that for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0_{\mathbb{C}}$ and $b = 0_{\mathbb{C}}$ and for every vector w of V there exist a, b such that $w = a \cdot u + b \cdot v$.

Let us consider A. The functor CRing(A) yielding a strict double loop structure is defined by:

(Def. 7) CRing(A) $= \langle \mathbb{C}^A, +_{\mathbb{C}^A}, \cdot_{\mathbb{C}^A}, \mathbf{1}_{\mathbb{C}^A}, \mathbf{0}_{\mathbb{C}^A}\rangle$.

Let us consider A. Observe that CRing(A) is non empty.

We now state two propositions:

(27) Let x, y, z be elements of CRing(A). Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x+0_{\mathrm{CRing}(A)} = x$ and there exists an element t of CRing(A) such that $x + t = 0_{\mathrm{CRing}(A)}$ and $x \cdot y = y \cdot x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{CRing}(A)} = x$ and $\mathbf{1}_{\mathrm{CRing}(A)} \cdot x = x$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$.

(28) CRing(A) is a commutative ring.

We introduce complex algebra structures which are extensions of double loop structure and CLS structure and are systems

⟨ a carrier, a multiplication, an addition, an external multiplication, a unity, a zero ⟩,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from $[\!:\mathbb{C}$, the carrier$:\!]$ into the carrier, and the unity and the zero are elements of the carrier.

Let us mention that there exists a complex algebra structure which is non empty.

Let us consider A. The functor CAlgebra(A) yielding a strict complex algebra structure is defined as follows:

(Def. 8) $\mathrm{CAlgebra}(A) = \langle \mathbb{C}^A, \cdot_{\mathbb{C}^A}, +_{\mathbb{C}^A}, \cdot^{\mathbb{C}}_{\mathbb{C}^A}, \mathbf{1}_{\mathbb{C}^A}, \mathbf{0}_{\mathbb{C}^A} \rangle$.

Let us consider A. Observe that CAlgebra(A) is non empty.

Next we state the proposition

(29) Let x, y, z be elements of CAlgebra(A) and given a, b. Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x + 0_{\mathrm{CAlgebra}(A)} = x$ and there exists an element t of CAlgebra(A) such that $x + t = 0_{\mathrm{CAlgebra}(A)}$ and $x \cdot y = y \cdot x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{CAlgebra}(A)} = x$ and $x \cdot (y+z) = x \cdot y + x \cdot z$ and $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ and $a \cdot (x+y) = a \cdot x + a \cdot y$ and $(a+b) \cdot x = a \cdot x + b \cdot x$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

Let I_1 be a non empty complex algebra structure. We say that I_1 is complex algebra-like if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let x, y, z be elements of I_1 and given a, b. Then $x \cdot \mathbf{1}_{(I_1)} = x$ and $x \cdot (y+z) = x \cdot y + x \cdot z$ and $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ and $a \cdot (x+y) = a \cdot x + a \cdot y$ and $(a+b) \cdot x = a \cdot x + b \cdot x$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

Let us note that there exists a non empty complex algebra structure which is strict, Abelian, add-associative, right zeroed, right complementable, commutative, associative, and complex algebra-like.

A complex algebra is an Abelian add-associative right zeroed right complementable commutative associative complex algebra-like non empty complex algebra structure.

One can prove the following proposition

(30) CAlgebra(A) is a complex algebra.

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