FORMALIZED MATHEMATICS
Volume 12, Number 3, 2004
University of Białystok

Lucas Numbers and Generalized Fibonacci Numbers

Piotr Wojtecki
University of Białystok

Adam Grabowski[1]
University of Białystok

Summary. The recursive definition of Fibonacci sequences [3] is a good starting point for various variants and generalizations. We can here point out e.g. Lucas (with 2 and 1 as opening values) and the so-called generalized Fibonacci numbers (starting with arbitrary integers a and b).

In this paper, we introduce Lucas and G-numbers and we state their basic properties analogous to those proven in [10] and [5].

MML Identifier: FIB_NUM3.

The papers [15], [14], [11], [2], [6], [1], [13], [12], [8], [9], [4], [7], [3], and [10] provide the notation and terminology for this paper.

1. Preliminaries

In this paper a, b, k, n denote natural numbers.

The following propositions are true:

(1) For every real number a and for every natural number n such that $a^n = 0$ holds $a = 0$.

(2) For every non negative real number a holds $\sqrt{a} \cdot \sqrt{a} = a$.

(3) For every non empty real number a holds $a^2 = (-a)^2$.

(4) For every non empty natural number k holds $(k -' 1) + 2 = (k + 2) -' 1$.

(5) $(a + b)^2 = a \cdot a + a \cdot b + a \cdot b + b \cdot b$.

(6) For every non empty real number a holds $(a^n)^2 = a^{2 \cdot n}$.

[1]This work has been partially supported by the CALCULEMUS grant HPRN-CT-2000-00102.

© 2004 University of Białystok
ISSN 1426–2630

(7) For all real numbers a, b holds $(a+b)\cdot(a-b) = a^2 - b^2$.

(8) For all non empty real numbers a, b holds $(a\cdot b)^n = a^n \cdot b^n$.

Let us mention that τ is positive and $\overline{\tau}$ is negative.

The following propositions are true:

(9) For every natural number n holds $\tau^n + \tau^{n+1} = \tau^{n+2}$.

(10) For every natural number n holds $\overline{\tau}^n + \overline{\tau}^{n+1} = \overline{\tau}^{n+2}$.

2. Lucas Numbers

Let n be a natural number. The functor $\mathrm{Luc}(n)$ yielding a natural number is defined by the condition (Def. 1).

(Def. 1) There exists a function L from $\mathbb{N}$ into $[\!:\mathbb{N}, \mathbb{N}:\!]$ such that $\mathrm{Luc}(n) = L(n)_{\mathbf{1}}$ and $L(0) = \langle 2, 1\rangle$ and for every natural number n holds $L(n+1) = \langle L(n)_{\mathbf{2}}, L(n)_{\mathbf{1}} + L(n)_{\mathbf{2}}\rangle$.

The following propositions are true:

(11) $\mathrm{Luc}(0) = 2$ and $\mathrm{Luc}(1) = 1$ and for every natural number n holds $\mathrm{Luc}(n+1+1) = \mathrm{Luc}(n) + \mathrm{Luc}(n+1)$.

(12) For every natural number n holds $\mathrm{Luc}(n+2) = \mathrm{Luc}(n) + \mathrm{Luc}(n+1)$.

(13) For every natural number n holds $\mathrm{Luc}(n+1)+\mathrm{Luc}(n+2) = \mathrm{Luc}(n+3)$.

(14) $\mathrm{Luc}(2) = 3$.

(15) $\mathrm{Luc}(3) = 4$.

(16) $\mathrm{Luc}(4) = 7$.

(17) For every natural number k holds $\mathrm{Luc}(k) \geqslant k$.

(18) For every non empty natural number m holds $\mathrm{Luc}(m+1) \geqslant \mathrm{Luc}(m)$.

Let n be a natural number. Note that $\mathrm{Luc}(n)$ is positive.

Next we state a number of propositions:

(19) For every natural number n holds $2\cdot\mathrm{Luc}(n+2) = \mathrm{Luc}(n) + \mathrm{Luc}(n+3)$.

(20) For every natural number n holds $\mathrm{Luc}(n+1) = \mathrm{Fib}(n) + \mathrm{Fib}(n+2)$.

(21) For every natural number n holds $\mathrm{Luc}(n) = \tau^n + \overline{\tau}^n$.

(22) For every natural number n holds $2\cdot\mathrm{Luc}(n)+\mathrm{Luc}(n+1) = 5\cdot\mathrm{Fib}(n+1)$.

(23) For every natural number n holds $\mathrm{Luc}(n+3) - 2\cdot\mathrm{Luc}(n) = 5\cdot\mathrm{Fib}(n)$.

(24) For every natural number n holds $\mathrm{Luc}(n) + \mathrm{Fib}(n) = 2\cdot\mathrm{Fib}(n+1)$.

(25) For every natural number n holds $3\cdot\mathrm{Fib}(n) + \mathrm{Luc}(n) = 2\cdot\mathrm{Fib}(n+2)$.

(26) For all natural numbers n, m holds $2\cdot\mathrm{Luc}(n+m) = \mathrm{Luc}(n)\cdot\mathrm{Luc}(m) + 5\cdot\mathrm{Fib}(n)\cdot\mathrm{Fib}(m)$.

(27) For every natural number n holds $\mathrm{Luc}(n+3)\cdot\mathrm{Luc}(n) = \mathrm{Luc}(n+2)^2 - \mathrm{Luc}(n+1)^2$.

(28) For every natural number n holds $\mathrm{Fib}(2\cdot n) = \mathrm{Fib}(n)\cdot\mathrm{Luc}(n)$.

(29) For every natural number n holds $2 \cdot \mathrm{Fib}(2 \cdot n+1) = \mathrm{Luc}(n+1) \cdot \mathrm{Fib}(n) + \mathrm{Luc}(n) \cdot \mathrm{Fib}(n+1)$.

(30) For every natural number n holds $5 \cdot \mathrm{Fib}(n)^2 - \mathrm{Luc}(n)^2 = 4 \cdot (-1)^{n+1}$.

(31) For every natural number n holds $\mathrm{Fib}(2 \cdot n + 1) = \mathrm{Fib}(n+1) \cdot \mathrm{Luc}(n+1) - \mathrm{Fib}(n) \cdot \mathrm{Luc}(n)$.

3. Generalized Fibonacci Numbers

Let a, b, n be natural numbers. The functor $\mathrm{GFib}(a, b, n)$ yielding a natural number is defined by the condition (Def. 2).

(Def. 2) There exists a function L from $\mathbb{N}$ into $[: \mathbb{N}, \mathbb{N} :]$ such that $\mathrm{GFib}(a, b, n) = L(n)_{\mathbf{1}}$ and $L(0) = \langle a, b \rangle$ and for every natural number n holds $L(n+1) = \langle L(n)_{\mathbf{2}}, L(n)_{\mathbf{1}} + L(n)_{\mathbf{2}} \rangle$.

Next we state a number of propositions:

(32) For all natural numbers a, b holds $\mathrm{GFib}(a, b, 0) = a$ and $\mathrm{GFib}(a, b, 1) = b$ and for every natural number n holds $\mathrm{GFib}(a, b, n+1+1) = \mathrm{GFib}(a, b, n) + \mathrm{GFib}(a, b, n + 1)$.

(33) $(\mathrm{GFib}(a, b, k + 1) + \mathrm{GFib}(a, b, k + 1 + 1))^2 = \mathrm{GFib}(a, b, k + 1)^2 + 2 \cdot \mathrm{GFib}(a, b, k + 1) \cdot \mathrm{GFib}(a, b, k + 1 + 1) + \mathrm{GFib}(a, b, k + 1 + 1)^2$.

(34) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) + \mathrm{GFib}(a, b, n + 1) = \mathrm{GFib}(a, b, n + 2)$.

(35) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n+1) + \mathrm{GFib}(a, b, n+2) = \mathrm{GFib}(a, b, n + 3)$.

(36) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n+2) + \mathrm{GFib}(a, b, n+3) = \mathrm{GFib}(a, b, n + 4)$.

(37) For every natural number n holds $\mathrm{GFib}(0, 1, n) = \mathrm{Fib}(n)$.

(38) For every natural number n holds $\mathrm{GFib}(2, 1, n) = \mathrm{Luc}(n)$.

(39) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) + \mathrm{GFib}(a, b, n + 3) = 2 \cdot \mathrm{GFib}(a, b, n + 2)$.

(40) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) + \mathrm{GFib}(a, b, n + 4) = 3 \cdot \mathrm{GFib}(a, b, n + 2)$.

(41) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n + 3) - \mathrm{GFib}(a, b, n) = 2 \cdot \mathrm{GFib}(a, b, n + 1)$.

(42) For all non empty natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) = \mathrm{GFib}(a, b, 0) \cdot \mathrm{Fib}(n -' 1) + \mathrm{GFib}(a, b, 1) \cdot \mathrm{Fib}(n)$.

(43) For all natural numbers n, m holds $\mathrm{Fib}(m) \cdot \mathrm{Luc}(n) + \mathrm{Luc}(m) \cdot \mathrm{Fib}(n) = \mathrm{GFib}(\mathrm{Fib}(0), \mathrm{Luc}(0), n + m)$.

(44) For every natural number n holds $\mathrm{Luc}(n) + \mathrm{Luc}(n + 3) = 2 \cdot \mathrm{Luc}(n + 2)$.

(45) For all natural numbers a, n holds $\mathrm{GFib}(a, a, n) = \frac{\mathrm{GFib}(a,a,0)}{2} \cdot (\mathrm{Fib}(n) + \mathrm{Luc}(n))$.

(46) For all natural numbers a, b, n holds $\mathrm{GFib}(b, a+b, n) = \mathrm{GFib}(a, b, n+1)$.

(47) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n+2) \cdot \mathrm{GFib}(a, b, n) - \mathrm{GFib}(a, b, n+1)^2 = (-1)^n \cdot (\mathrm{GFib}(a, b, 2)^2 - \mathrm{GFib}(a, b, 1) \cdot \mathrm{GFib}(a, b, 3))$.

(48) For all natural numbers a, b, k, n holds $\mathrm{GFib}(\mathrm{GFib}(a, b, k), \mathrm{GFib}(a, b, k+1), n) = \mathrm{GFib}(a, b, n+k)$.

(49) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n+1) = a \cdot \mathrm{Fib}(n) + b \cdot \mathrm{Fib}(n+1)$.

(50) For all natural numbers a, b, n holds $\mathrm{GFib}(0, b, n) = b \cdot \mathrm{Fib}(n)$.

(51) For all natural numbers a, b, n holds $\mathrm{GFib}(a, 0, n+1) = a \cdot \mathrm{Fib}(n)$.

(52) For all natural numbers a, b, c, d, n holds $\mathrm{GFib}(a, b, n) + \mathrm{GFib}(c, d, n) = \mathrm{GFib}(a+c, b+d, n)$.

(53) For all natural numbers a, b, k, n holds $\mathrm{GFib}(k \cdot a, k \cdot b, n) = k \cdot \mathrm{GFib}(a, b, n)$.

(54) For all natural numbers a, b, n holds $\mathrm{GFib}(a, b, n) = \frac{(a \cdot -\overline{\tau} + b) \cdot \tau^n + (a \cdot \tau - b) \cdot \overline{\tau}^n}{\sqrt{5}}$.

(55) For all natural numbers a, n holds $\mathrm{GFib}(2 \cdot a + 1, 2 \cdot a + 1, n + 1) = (2 \cdot a + 1) \cdot \mathrm{Fib}(n + 2)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Grzegorz Bancerek and Piotr Rudnicki. Two programs for **scm**. Part I - preliminaries. *Formalized Mathematics*, 4(**1**):69–72, 1993.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Magdalena Jastrzębska and Adam Grabowski. Some properties of Fibonacci numbers. *Formalized Mathematics*, 12(**3**):307–313, 2004.

[6] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[7] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.

[8] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.

[9] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.

[10] Robert M. Solovay. Fibonacci numbers. *Formalized Mathematics*, 10(**2**):81–83, 2002.

[11] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[12] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[13] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[14] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received May 24, 2004