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Continuous Functions on Real and Complex Normed Linear Spaces

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Summary. This article is an extension of [18].

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The notation and terminology used here are introduced in the following papers: [25], [28], [29], [4], [30], [6], [14], [5], [2], [24], [10], [26], [27], [19], [15], [12], [13], [11], [31], [20], [3], [1], [16], [21], [17], [23], [7], [8], [22], [18], and [9].

For simplicity, we use the following convention: n denotes a natural number, r, s denote real numbers, z denotes a complex number, C_1, C_2, C_3 denote complex normed spaces, and R_1 denotes a real normed space.

Let C_4 be a complex linear space and let s_1 be a sequence of C_4. The functor $-s_1$ yields a sequence of C_4 and is defined by:

(Def. 1) For every n holds $(-s_1)(n) = -s_1(n)$.

The following propositions are true:

(1) For all sequences s_2, s_3 of C_1 holds $s_2 - s_3 = s_2 + -s_3$.

(2) For every sequence s_1 of C_1 holds $-s_1 = (-1_{\mathbb{C}}) \cdot s_1$.

Let us consider C_2, C_3 and let f be a partial function from C_2 to C_3. The functor $\|f\|$ yielding a partial function from the carrier of C_2 to $\mathbb{R}$ is defined by:

(Def. 2) $\mathrm{dom}\|f\| = \mathrm{dom}\, f$ and for every point c of C_2 such that $c \in \mathrm{dom}\|f\|$ holds $\|f\|(c) = \|f_c\|$.

Let us consider C_1, R_1 and let f be a partial function from C_1 to R_1. The functor $\|f\|$ yielding a partial function from the carrier of C_1 to $\mathbb{R}$ is defined as follows:

(Def. 3) $\mathrm{dom}\|f\| = \mathrm{dom}\, f$ and for every point c of C_1 such that $c \in \mathrm{dom}\|f\|$ holds $\|f\|(c) = \|f_c\|$.

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Let us consider R_1, C_1 and let f be a partial function from R_1 to C_1. The functor $\|f\|$ yielding a partial function from the carrier of R_1 to $\mathbb{R}$ is defined by:

(Def. 4) $\mathrm{dom}\|f\| = \mathrm{dom}\, f$ and for every point c of R_1 such that $c \in \mathrm{dom}\|f\|$ holds $\|f\|(c) = \|f_c\|$.

Let us consider C_1 and let x_0 be a point of C_1. A subset of C_1 is called a neighbourhood of x_0 if:

(Def. 5) There exists a real number g such that $0 < g$ and $\{y; y$ ranges over points of C_1: $\|y - x_0\| < g\} \subseteq$ it.

Next we state two propositions:

(3) Let x_0 be a point of C_1 and g be a real number. If $0 < g$, then $\{y; y$ ranges over points of C_1: $\|y - x_0\| < g\}$ is a neighbourhood of x_0.

(4) For every point x_0 of C_1 and for every neighbourhood N of x_0 holds $x_0 \in N$.

Let us consider C_1 and let X be a subset of C_1. We say that X is compact if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let s_4 be a sequence of C_1. Suppose $\mathrm{rng}\, s_4 \subseteq X$. Then there exists a sequence s_5 of C_1 such that s_5 is a subsequence of s_4 and convergent and $\lim s_5 \in X$.

Let us consider C_1 and let X be a subset of C_1. We say that X is closed if and only if:

(Def. 7) For every sequence s_4 of C_1 such that $\mathrm{rng}\, s_4 \subseteq X$ and s_4 is convergent holds $\lim s_4 \in X$.

Let us consider C_1 and let X be a subset of C_1. We say that X is open if and only if:

(Def. 8) X^{c} is closed.

Let us consider C_2, C_3, let f be a partial function from C_2 to C_3, and let s_1 be a sequence of C_2. Let us assume that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$. The functor $f \cdot s_1$ yields a sequence of C_3 and is defined by:

(Def. 9) $f \cdot s_1 = (f$ **qua** function$)\ \cdot (s_1)$.

Let us consider C_1, R_1, let f be a partial function from C_1 to R_1, and let s_1 be a sequence of C_1. Let us assume that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$. The functor $f \cdot s_1$ yielding a sequence of R_1 is defined by:

(Def. 10) $f \cdot s_1 = (f$ **qua** function$)\ \cdot (s_1)$.

Let us consider C_1, R_1, let f be a partial function from R_1 to C_1, and let s_1 be a sequence of R_1. Let us assume that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$. The functor $f \cdot s_1$ yields a sequence of C_1 and is defined by:

(Def. 11) $f \cdot s_1 = (f$ **qua** function$)\ \cdot (s_1)$.

Let us consider C_1, let f be a partial function from the carrier of C_1 to $\mathbb{C}$, and let s_1 be a sequence of C_1. Let us assume that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$. The functor

$f \cdot s_1$ yields a complex sequence and is defined as follows:

(Def. 12) $f \cdot s_1 = (f$ **qua** function$) \cdot (s_1)$.

Let us consider R_1, let f be a partial function from the carrier of R_1 to $\mathbb{C}$, and let s_1 be a sequence of R_1. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$. The functor $f \cdot s_1$ yielding a complex sequence is defined by:

(Def. 13) $f \cdot s_1 = (f$ **qua** function$) \cdot (s_1)$.

Let us consider C_1, let f be a partial function from the carrier of C_1 to $\mathbb{R}$, and let s_1 be a sequence of C_1. Let us assume that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$. The functor $f \cdot s_1$ yielding a sequence of real numbers is defined as follows:

(Def. 14) $f \cdot s_1 = (f$ **qua** function$) \cdot (s_1)$.

Let us consider C_2, C_3, let f be a partial function from C_2 to C_3, and let x_0 be a point of C_2. We say that f is continuous in x_0 if and only if the conditions (Def. 15) are satisfied.

(Def. 15)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of C_2 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider C_1, R_1, let f be a partial function from C_1 to R_1, and let x_0 be a point of C_1. We say that f is continuous in x_0 if and only if the conditions (Def. 16) are satisfied.

(Def. 16)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of C_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider R_1, let us consider C_1, let f be a partial function from R_1 to C_1, and let x_0 be a point of R_1. We say that f is continuous in x_0 if and only if the conditions (Def. 17) are satisfied.

(Def. 17)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of R_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider C_1, let f be a partial function from the carrier of C_1 to $\mathbb{C}$, and let x_0 be a point of C_1. We say that f is continuous in x_0 if and only if the conditions (Def. 18) are satisfied.

(Def. 18)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of C_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider C_1, let f be a partial function from the carrier of C_1 to $\mathbb{R}$, and let x_0 be a point of C_1. We say that f is continuous in x_0 if and only if the conditions (Def. 19) are satisfied.

(Def. 19)(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every sequence s_1 of C_1 such that $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

Let us consider R_1, let f be a partial function from the carrier of R_1 to $\mathbb{C}$, and let x_0 be a point of R_1. We say that f is continuous in x_0 if and only if the conditions (Def. 20) are satisfied.

(Def. 20)(i) $x_0 \in \mathrm{dom}\, f$, and

(ii) for every sequence s_1 of R_1 such that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f$ and s_1 is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim(f \cdot s_1)$.

The following propositions are true:

(5) For every sequence s_1 of C_2 and for every partial function h from C_2 to C_3 such that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$ holds $s_1(n) \in \mathrm{dom}\, h$.

(6) For every sequence s_1 of C_1 and for every partial function h from C_1 to R_1 such that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$ holds $s_1(n) \in \mathrm{dom}\, h$.

(7) For every sequence s_1 of R_1 and for every partial function h from R_1 to C_1 such that $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, h$ holds $s_1(n) \in \mathrm{dom}\, h$.

(8) For every sequence s_1 of C_1 and for every set x holds $x \in \mathrm{rng}\, s_1$ iff there exists n such that $x = s_1(n)$.

(9) For all sequences s_1, s_2 of C_1 such that s_2 is a subsequence of s_1 holds $\mathrm{rng}\, s_2 \subseteq \mathrm{rng}\, s_1$.

(10) Let f be a partial function from C_2 to C_3 and C_5 be a sequence of C_2. If $\mathrm{rng}\, C_5 \subseteq \mathrm{dom}\, f$, then for every n holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(11) Let f be a partial function from C_1 to R_1 and C_5 be a sequence of C_1. If $\mathrm{rng}\, C_5 \subseteq \mathrm{dom}\, f$, then for every n holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(12) Let f be a partial function from R_1 to C_1 and R_2 be a sequence of R_1. If $\mathrm{rng}\, R_2 \subseteq \mathrm{dom}\, f$, then for every n holds $(f \cdot R_2)(n) = f_{R_2(n)}$.

(13) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$ and C_5 be a sequence of C_1. If $\mathrm{rng}\, C_5 \subseteq \mathrm{dom}\, f$, then for every n holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(14) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$ and C_5 be a sequence of C_1. If $\mathrm{rng}\, C_5 \subseteq \mathrm{dom}\, f$, then for every n holds $(f \cdot C_5)(n) = f_{C_5(n)}$.

(15) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$ and R_2 be a sequence of R_1. If $\mathrm{rng}\, R_2 \subseteq \mathrm{dom}\, f$, then for every n holds $(f \cdot R_2)(n) = f_{R_2(n)}$.

(16) Let h be a partial function from C_2 to C_3, C_5 be a sequence of C_2, and N_1 be an increasing sequence of naturals. If $\mathrm{rng}\, C_5 \subseteq \mathrm{dom}\, h$, then $(h \cdot C_5) \cdot N_1 = h \cdot (C_5 \cdot N_1)$.

(17) Let h be a partial function from C_1 to R_1, C_6 be a sequence of C_1, and N_1 be an increasing sequence of naturals. If $\mathrm{rng}\, C_6 \subseteq \mathrm{dom}\, h$, then $(h \cdot C_6) \cdot N_1 = h \cdot (C_6 \cdot N_1)$.

(18) Let h be a partial function from R_1 to C_1, R_3 be a sequence of R_1,

and N_1 be an increasing sequence of naturals. If $\operatorname{rng} R_3 \subseteq \operatorname{dom} h$, then $(h \cdot R_3) \cdot N_1 = h \cdot (R_3 \cdot N_1)$.

(19) Let h be a partial function from the carrier of C_1 to $\mathbb{C}$, C_6 be a sequence of C_1, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} C_6 \subseteq \operatorname{dom} h$, then $(h \cdot C_6) \cdot N_1 = h \cdot (C_6 \cdot N_1)$.

(20) Let h be a partial function from the carrier of C_1 to $\mathbb{R}$, C_6 be a sequence of C_1, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} C_6 \subseteq \operatorname{dom} h$, then $(h \cdot C_6) \cdot N_1 = h \cdot (C_6 \cdot N_1)$.

(21) Let h be a partial function from the carrier of R_1 to $\mathbb{C}$, R_3 be a sequence of R_1, and N_1 be an increasing sequence of naturals. If $\operatorname{rng} R_3 \subseteq \operatorname{dom} h$, then $(h \cdot R_3) \cdot N_1 = h \cdot (R_3 \cdot N_1)$.

(22) Let h be a partial function from C_2 to C_3 and C_7, C_8 be sequences of C_2. If $\operatorname{rng} C_7 \subseteq \operatorname{dom} h$ and C_8 is a subsequence of C_7, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(23) Let h be a partial function from C_1 to R_1 and C_7, C_8 be sequences of C_1. If $\operatorname{rng} C_7 \subseteq \operatorname{dom} h$ and C_8 is a subsequence of C_7, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(24) Let h be a partial function from R_1 to C_1 and R_4, R_5 be sequences of R_1. If $\operatorname{rng} R_4 \subseteq \operatorname{dom} h$ and R_5 is a subsequence of R_4, then $h \cdot R_5$ is a subsequence of $h \cdot R_4$.

(25) Let s_1 be a complex sequence, n be a natural number, and N_2 be an increasing sequence of naturals. Then $(s_1 \cdot N_2)(n) = s_1(N_2(n))$.

(26) Let h be a partial function from the carrier of C_1 to $\mathbb{C}$ and C_7, C_8 be sequences of C_1. If $\operatorname{rng} C_7 \subseteq \operatorname{dom} h$ and C_8 is a subsequence of C_7, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(27) Let h be a partial function from the carrier of C_1 to $\mathbb{R}$ and C_7, C_8 be sequences of C_1. If $\operatorname{rng} C_7 \subseteq \operatorname{dom} h$ and C_8 is a subsequence of C_7, then $h \cdot C_8$ is a subsequence of $h \cdot C_7$.

(28) Let h be a partial function from the carrier of R_1 to $\mathbb{C}$ and R_4, R_5 be sequences of R_1. If $\operatorname{rng} R_4 \subseteq \operatorname{dom} h$ and R_5 is a subsequence of R_4, then $h \cdot R_5$ is a subsequence of $h \cdot R_4$.

(29) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_2 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(30) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(31) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of R_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(32) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$ and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(33) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$ and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(34) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$ and x_0 be a point of R_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every r such that $0 < r$ there exists s such that $0 < s$ and for every point x_1 of R_1 such that $x_1 \in \operatorname{dom} f$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(35) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that for every point x_1 of C_2 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_3$.

(36) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that for every point x_1 of C_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_3$.

(37) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that for every point x_1 of R_1 such that $x_1 \in \operatorname{dom} f$ and $x_1 \in N$ holds $f_{x_1} \in N_3$.

(38) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_3$.

(39) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_3$.

(40) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. Then f is continuous in x_0 if and only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every neighbourhood N_3 of f_{x_0} there exists a neighbourhood N of x_0 such that $f^\circ N \subseteq N_3$.

(41) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. Suppose $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$. Then f is continuous in x_0.

(42) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. Suppose $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$. Then f is continuous in x_0.

(43) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. Suppose $x_0 \in \operatorname{dom} f$ and there exists a neighbourhood N of x_0 such that $\operatorname{dom} f \cap N = \{x_0\}$. Then f is continuous in x_0.

(44) Let h_1, h_2 be partial functions from C_2 to C_3 and s_1 be a sequence of C_2. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(45) Let h_1, h_2 be partial functions from C_1 to R_1 and s_1 be a sequence of C_1. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(46) Let h_1, h_2 be partial functions from R_1 to C_1 and s_1 be a sequence of R_1. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h_1 \cap \operatorname{dom} h_2$, then $(h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1$ and $(h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1$.

(47) Let h be a partial function from C_2 to C_3, s_1 be a sequence of C_2, and z be a complex number. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(z\, h) \cdot s_1 = z \cdot (h \cdot s_1)$.

(48) Let h be a partial function from C_1 to R_1, s_1 be a sequence of C_1, and r be a real number. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(r\,h) \cdot s_1 = r \cdot (h \cdot s_1)$.

(49) Let h be a partial function from R_1 to C_1, s_1 be a sequence of R_1, and z be a complex number. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $(z\,h) \cdot s_1 = z \cdot (h \cdot s_1)$.

(50) Let h be a partial function from C_2 to C_3 and s_1 be a sequence of C_2. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $\|h \cdot s_1\| = \|h\| \cdot s_1$ and $-h \cdot s_1 = (-h) \cdot s_1$.

(51) Let h be a partial function from C_1 to R_1 and s_1 be a sequence of C_1. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $\|h \cdot s_1\| = \|h\| \cdot s_1$ and $-h \cdot s_1 = (-h) \cdot s_1$.

(52) Let h be a partial function from R_1 to C_1 and s_1 be a sequence of R_1. If $\operatorname{rng} s_1 \subseteq \operatorname{dom} h$, then $\|h \cdot s_1\| = \|h\| \cdot s_1$ and $-h \cdot s_1 = (-h) \cdot s_1$.

(53) Let f_1, f_2 be partial functions from C_2 to C_3 and x_0 be a point of C_2. Suppose f_1 is continuous in x_0 and f_2 is continuous in x_0. Then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(54) Let f_1, f_2 be partial functions from C_1 to R_1 and x_0 be a point of C_1. Suppose f_1 is continuous in x_0 and f_2 is continuous in x_0. Then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(55) Let f_1, f_2 be partial functions from R_1 to C_1 and x_0 be a point of R_1. Suppose f_1 is continuous in x_0 and f_2 is continuous in x_0. Then $f_1 + f_2$ is continuous in x_0 and $f_1 - f_2$ is continuous in x_0.

(56) Let f be a partial function from C_2 to C_3, x_0 be a point of C_2, and z be a complex number. If f is continuous in x_0, then $z\,f$ is continuous in x_0.

(57) Let f be a partial function from C_1 to R_1, x_0 be a point of C_1, and r be a real number. If f is continuous in x_0, then $r\,f$ is continuous in x_0.

(58) Let f be a partial function from R_1 to C_1, x_0 be a point of R_1, and z be a complex number. If f is continuous in x_0, then $z\,f$ is continuous in x_0.

(59) Let f be a partial function from C_2 to C_3 and x_0 be a point of C_2. If f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

(60) Let f be a partial function from C_1 to R_1 and x_0 be a point of C_1. If f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

(61) Let f be a partial function from R_1 to C_1 and x_0 be a point of R_1. If f is continuous in x_0, then $\|f\|$ is continuous in x_0 and $-f$ is continuous in x_0.

Let C_2, C_3 be complex normed spaces, let f be a partial function from C_2 to C_3, and let X be a set. We say that f is continuous on X if and only if:

(Def. 21) $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_2 such that $x_0 \in X$ holds $f{\restriction}X$ is continuous in x_0.

Let C_1 be a complex normed space, let R_1 be a real normed space, let f be a partial function from C_1 to R_1, and let X be a set. We say that f is continuous on X if and only if:

(Def. 22) $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

Let R_1 be a real normed space, let C_1 be a complex normed space, let g be a partial function from R_1 to C_1, and let X be a set. We say that g is continuous on X if and only if:

(Def. 23) $X \subseteq \operatorname{dom} g$ and for every point x_0 of R_1 such that $x_0 \in X$ holds $g \restriction X$ is continuous in x_0.

Let C_1 be a complex normed space, let f be a partial function from the carrier of C_1 to $\mathbb{C}$, and let X be a set. We say that f is continuous on X if and only if:

(Def. 24) $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

Let C_1 be a complex normed space, let f be a partial function from the carrier of C_1 to $\mathbb{R}$, and let X be a set. We say that f is continuous on X if and only if:

(Def. 25) $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

Let R_1 be a real normed space, let f be a partial function from the carrier of R_1 to $\mathbb{C}$, and let X be a set. We say that f is continuous on X if and only if:

(Def. 26) $X \subseteq \operatorname{dom} f$ and for every point x_0 of R_1 such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

In the sequel X, X_1 denote sets.

The following propositions are true:

(62) Let f be a partial function from C_2 to C_3. Then f is continuous on X if and only if the following conditions are satisfied:
- (i) $X \subseteq \operatorname{dom} f$, and
- (ii) for every sequence s_4 of C_2 such that $\operatorname{rng} s_4 \subseteq X$ and s_4 is convergent and $\lim s_4 \in X$ holds $f \cdot s_4$ is convergent and $f_{\lim s_4} = \lim(f \cdot s_4)$.

(63) Let f be a partial function from C_1 to R_1. Then f is continuous on X if and only if the following conditions are satisfied:
- (i) $X \subseteq \operatorname{dom} f$, and
- (ii) for every sequence s_4 of C_1 such that $\operatorname{rng} s_4 \subseteq X$ and s_4 is convergent and $\lim s_4 \in X$ holds $f \cdot s_4$ is convergent and $f_{\lim s_4} = \lim(f \cdot s_4)$.

(64) Let f be a partial function from R_1 to C_1. Then f is continuous on X if and only if the following conditions are satisfied:
- (i) $X \subseteq \operatorname{dom} f$, and
- (ii) for every sequence s_4 of R_1 such that $\operatorname{rng} s_4 \subseteq X$ and s_4 is convergent and $\lim s_4 \in X$ holds $f \cdot s_4$ is convergent and $f_{\lim s_4} = \lim(f \cdot s_4)$.

(65) Let f be a partial function from C_2 to C_3. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every point x_0 of C_2 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_2 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(66) Let f be a partial function from C_1 to R_1. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every point x_0 of C_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(67) Let f be a partial function from R_1 to C_1. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every point x_0 of R_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of R_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.

(68) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every point x_0 of C_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(69) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every point x_0 of C_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of C_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(70) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every point x_0 of R_1 and for every r such that $x_0 \in X$ and $0 < r$ there exists s such that $0 < s$ and for every point x_1 of R_1 such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f_{x_1} - f_{x_0}| < r$.

(71) For every partial function f from C_2 to C_3 holds f is continuous on X iff $f{\restriction}X$ is continuous on X.

(72) For every partial function f from C_1 to R_1 holds f is continuous on X iff $f{\restriction}X$ is continuous on X.

(73) For every partial function f from R_1 to C_1 holds f is continuous on X iff $f{\restriction}X$ is continuous on X.

(74) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. Then f is continuous on X if and only if $f{\restriction}X$ is continuous on X.

(75) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Then f is continuous on X if and only if $f{\restriction}X$ is continuous on X.

(76) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. Then f is continuous on X if and only if $f{\restriction}X$ is continuous on X.

(77) For every partial function f from C_2 to C_3 such that f is continuous on X and $X_1 \subseteq X$ holds f is continuous on X_1.

(78) For every partial function f from C_1 to R_1 such that f is continuous on X and $X_1 \subseteq X$ holds f is continuous on X_1.

(79) For every partial function f from R_1 to C_1 such that f is continuous on X and $X_1 \subseteq X$ holds f is continuous on X_1.

(80) For every partial function f from C_2 to C_3 and for every point x_0 of C_2 such that $x_0 \in \operatorname{dom} f$ holds f is continuous on $\{x_0\}$.

(81) For every partial function f from C_1 to R_1 and for every point x_0 of C_1 such that $x_0 \in \operatorname{dom} f$ holds f is continuous on $\{x_0\}$.

(82) For every partial function f from R_1 to C_1 and for every point x_0 of R_1 such that $x_0 \in \operatorname{dom} f$ holds f is continuous on $\{x_0\}$.

(83) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is continuous on X and f_2 is continuous on X. Then $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X.

(84) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is continuous on X and f_2 is continuous on X. Then $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X.

(85) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is continuous on X and f_2 is continuous on X. Then $f_1 + f_2$ is continuous on X and $f_1 - f_2$ is continuous on X.

(86) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

(87) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

(88) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is continuous on X and f_2 is continuous on X_1. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

(89) For every partial function f from C_2 to C_3 such that f is continuous on

X holds $z\,f$ is continuous on X.

(90) For every partial function f from C_1 to R_1 such that f is continuous on X holds $r\,f$ is continuous on X.

(91) For every partial function f from R_1 to C_1 such that f is continuous on X holds $z\,f$ is continuous on X.

(92) Let f be a partial function from C_2 to C_3. If f is continuous on X, then $\|f\|$ is continuous on X and $-f$ is continuous on X.

(93) Let f be a partial function from C_1 to R_1. If f is continuous on X, then $\|f\|$ is continuous on X and $-f$ is continuous on X.

(94) Let f be a partial function from R_1 to C_1. If f is continuous on X, then $\|f\|$ is continuous on X and $-f$ is continuous on X.

(95) Let f be a partial function from C_2 to C_3. Suppose f is total and for all points x_1, x_2 of C_2 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists a point x_0 of C_2 such that f is continuous in x_0. Then f is continuous on the carrier of C_2.

(96) Let f be a partial function from C_1 to R_1. Suppose f is total and for all points x_1, x_2 of C_1 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists a point x_0 of C_1 such that f is continuous in x_0. Then f is continuous on the carrier of C_1.

(97) Let f be a partial function from R_1 to C_1. Suppose f is total and for all points x_1, x_2 of R_1 holds $f_{x_1+x_2} = f_{x_1} + f_{x_2}$ and there exists a point x_0 of R_1 such that f is continuous in x_0. Then f is continuous on the carrier of R_1.

(98) For every partial function f from C_2 to C_3 such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(99) For every partial function f from C_1 to R_1 such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(100) For every partial function f from R_1 to C_1 such that $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$ holds $\operatorname{rng} f$ is compact.

(101) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. If $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$, then $\operatorname{rng} f$ is compact.

(102) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. If $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$, then $\operatorname{rng} f$ is compact.

(103) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. If $\operatorname{dom} f$ is compact and f is continuous on $\operatorname{dom} f$, then $\operatorname{rng} f$ is compact.

(104) Let Y be a subset of C_2 and f be a partial function from C_2 to C_3. If $Y \subseteq \operatorname{dom} f$ and Y is compact and f is continuous on Y, then $f^\circ Y$ is compact.

(105) Let Y be a subset of C_1 and f be a partial function from C_1 to R_1.

If $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y, then $f^\circ Y$ is compact.

(106) Let Y be a subset of R_1 and f be a partial function from R_1 to C_1. If $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y, then $f^\circ Y$ is compact.

(107) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Suppose $\mathrm{dom}\, f \neq \emptyset$ and $\mathrm{dom}\, f$ is compact and f is continuous on $\mathrm{dom}\, f$. Then there exist points x_1, x_2 of C_1 such that $x_1 \in \mathrm{dom}\, f$ and $x_2 \in \mathrm{dom}\, f$ and $f_{x_1} = \sup \mathrm{rng}\, f$ and $f_{x_2} = \inf \mathrm{rng}\, f$.

(108) Let f be a partial function from C_2 to C_3. Suppose $\mathrm{dom}\, f \neq \emptyset$ and $\mathrm{dom}\, f$ is compact and f is continuous on $\mathrm{dom}\, f$. Then there exist points x_1, x_2 of C_2 such that $x_1 \in \mathrm{dom}\, f$ and $x_2 \in \mathrm{dom}\, f$ and $\|f\|_{x_1} = \sup \mathrm{rng} \|f\|$ and $\|f\|_{x_2} = \inf \mathrm{rng} \|f\|$.

(109) Let f be a partial function from C_1 to R_1. Suppose $\mathrm{dom}\, f \neq \emptyset$ and $\mathrm{dom}\, f$ is compact and f is continuous on $\mathrm{dom}\, f$. Then there exist points x_1, x_2 of C_1 such that $x_1 \in \mathrm{dom}\, f$ and $x_2 \in \mathrm{dom}\, f$ and $\|f\|_{x_1} = \sup \mathrm{rng} \|f\|$ and $\|f\|_{x_2} = \inf \mathrm{rng} \|f\|$.

(110) Let f be a partial function from R_1 to C_1. Suppose $\mathrm{dom}\, f \neq \emptyset$ and $\mathrm{dom}\, f$ is compact and f is continuous on $\mathrm{dom}\, f$. Then there exist points x_1, x_2 of R_1 such that $x_1 \in \mathrm{dom}\, f$ and $x_2 \in \mathrm{dom}\, f$ and $\|f\|_{x_1} = \sup \mathrm{rng} \|f\|$ and $\|f\|_{x_2} = \inf \mathrm{rng} \|f\|$.

(111) For every partial function f from C_2 to C_3 holds $\|f\| \restriction X = \|f \restriction X\|$.

(112) For every partial function f from C_1 to R_1 holds $\|f\| \restriction X = \|f \restriction X\|$.

(113) For every partial function f from R_1 to C_1 holds $\|f\| \restriction X = \|f \restriction X\|$.

(114) Let f be a partial function from C_2 to C_3 and Y be a subset of C_2. Suppose $Y \neq \emptyset$ and $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y. Then there exist points x_1, x_2 of C_2 such that $x_1 \in Y$ and $x_2 \in Y$ and $\|f\|_{x_1} = \sup(\|f\|^\circ Y)$ and $\|f\|_{x_2} = \inf(\|f\|^\circ Y)$.

(115) Let f be a partial function from C_1 to R_1 and Y be a subset of C_1. Suppose $Y \neq \emptyset$ and $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y. Then there exist points x_1, x_2 of C_1 such that $x_1 \in Y$ and $x_2 \in Y$ and $\|f\|_{x_1} = \sup(\|f\|^\circ Y)$ and $\|f\|_{x_2} = \inf(\|f\|^\circ Y)$.

(116) Let f be a partial function from R_1 to C_1 and Y be a subset of R_1. Suppose $Y \neq \emptyset$ and $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y. Then there exist points x_1, x_2 of R_1 such that $x_1 \in Y$ and $x_2 \in Y$ and $\|f\|_{x_1} = \sup(\|f\|^\circ Y)$ and $\|f\|_{x_2} = \inf(\|f\|^\circ Y)$.

(117) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$ and Y be a subset of C_1. Suppose $Y \neq \emptyset$ and $Y \subseteq \mathrm{dom}\, f$ and Y is compact and f is continuous on Y. Then there exist points x_1, x_2 of C_1 such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \sup(f^\circ Y)$ and $f_{x_2} = \inf(f^\circ Y)$.

Let C_2, C_3 be complex normed spaces, let X be a set, and let f be a partial function from C_2 to C_3. We say that f is Lipschitzian on X if and only if:

(Def. 27) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of C_2 such that $x_1 \in X$ and $x_2 \in X$ holds $\|f_{x_1} - f_{x_2}\| \leqslant r \cdot \|x_1 - x_2\|$.

Let C_1 be a complex normed space, let R_1 be a real normed space, let X be a set, and let f be a partial function from C_1 to R_1. We say that f is Lipschitzian on X if and only if:

(Def. 28) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ holds $\|f_{x_1} - f_{x_2}\| \leqslant r \cdot \|x_1 - x_2\|$.

Let R_1 be a real normed space, let C_1 be a complex normed space, let X be a set, and let f be a partial function from R_1 to C_1. We say that f is Lipschitzian on X if and only if:

(Def. 29) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of R_1 such that $x_1 \in X$ and $x_2 \in X$ holds $\|f_{x_1} - f_{x_2}\| \leqslant r \cdot \|x_1 - x_2\|$.

Let C_1 be a complex normed space, let X be a set, and let f be a partial function from the carrier of C_1 to $\mathbb{C}$. We say that f is Lipschitzian on X if and only if:

(Def. 30) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ holds $|f_{x_1} - f_{x_2}| \leqslant r \cdot \|x_1 - x_2\|$.

Let C_1 be a complex normed space, let X be a set, and let f be a partial function from the carrier of C_1 to $\mathbb{R}$. We say that f is Lipschitzian on X if and only if:

(Def. 31) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of C_1 such that $x_1 \in X$ and $x_2 \in X$ holds $|f_{x_1} - f_{x_2}| \leqslant r \cdot \|x_1 - x_2\|$.

Let R_1 be a real normed space, let X be a set, and let f be a partial function from the carrier of R_1 to $\mathbb{C}$. We say that f is Lipschitzian on X if and only if:

(Def. 32) $X \subseteq \operatorname{dom} f$ and there exists r such that $0 < r$ and for all points x_1, x_2 of R_1 such that $x_1 \in X$ and $x_2 \in X$ holds $|f_{x_1} - f_{x_2}| \leqslant r \cdot \|x_1 - x_2\|$.

Next we state a number of propositions:

(118) For every partial function f from C_2 to C_3 such that f is Lipschitzian on X and $X_1 \subseteq X$ holds f is Lipschitzian on X_1.

(119) For every partial function f from C_1 to R_1 such that f is Lipschitzian on X and $X_1 \subseteq X$ holds f is Lipschitzian on X_1.

(120) For every partial function f from R_1 to C_1 such that f is Lipschitzian on X and $X_1 \subseteq X$ holds f is Lipschitzian on X_1.

(121) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(122) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(123) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(124) Let f_1, f_2 be partial functions from C_2 to C_3. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(125) Let f_1, f_2 be partial functions from C_1 to R_1. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(126) Let f_1, f_2 be partial functions from R_1 to C_1. Suppose f_1 is Lipschitzian on X and f_2 is Lipschitzian on X_1. Then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(127) For every partial function f from C_2 to C_3 such that f is Lipschitzian on X holds $z\,f$ is Lipschitzian on X.

(128) For every partial function f from C_1 to R_1 such that f is Lipschitzian on X holds $r\,f$ is Lipschitzian on X.

(129) For every partial function f from R_1 to C_1 such that f is Lipschitzian on X holds $z\,f$ is Lipschitzian on X.

(130) Let f be a partial function from C_2 to C_3. Suppose f is Lipschitzian on X. Then $-f$ is Lipschitzian on X and $\|f\|$ is Lipschitzian on X.

(131) Let f be a partial function from C_1 to R_1. Suppose f is Lipschitzian on X. Then $-f$ is Lipschitzian on X and $\|f\|$ is Lipschitzian on X.

(132) Let f be a partial function from R_1 to C_1. Suppose f is Lipschitzian on X. Then $-f$ is Lipschitzian on X and $\|f\|$ is Lipschitzian on X.

(133) Let X be a set and f be a partial function from C_2 to C_3. If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is Lipschitzian on X.

(134) Let X be a set and f be a partial function from C_1 to R_1. If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is Lipschitzian on X.

(135) Let X be a set and f be a partial function from R_1 to C_1. If $X \subseteq \mathrm{dom}\, f$ and f is a constant on X, then f is Lipschitzian on X.

(136) For every subset Y of C_1 holds id_Y is Lipschitzian on Y.

(137) For every partial function f from C_2 to C_3 such that f is Lipschitzian on X holds f is continuous on X.

(138) For every partial function f from C_1 to R_1 such that f is Lipschitzian on X holds f is continuous on X.

(139) For every partial function f from R_1 to C_1 such that f is Lipschitzian on X holds f is continuous on X.

(140) Let f be a partial function from the carrier of C_1 to $\mathbb{C}$. If f is Lipschitzian on X, then f is continuous on X.

(141) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. If f is Lipschitzian on X, then f is continuous on X.

(142) Let f be a partial function from the carrier of R_1 to $\mathbb{C}$. If f is Lipschitzian on X, then f is continuous on X.

(143) For every partial function f from C_2 to C_3 such that there exists a point r of C_3 such that $\operatorname{rng} f = \{r\}$ holds f is continuous on $\operatorname{dom} f$.

(144) For every partial function f from C_1 to R_1 such that there exists a point r of R_1 such that $\operatorname{rng} f = \{r\}$ holds f is continuous on $\operatorname{dom} f$.

(145) For every partial function f from R_1 to C_1 such that there exists a point r of C_1 such that $\operatorname{rng} f = \{r\}$ holds f is continuous on $\operatorname{dom} f$.

(146) For every partial function f from C_2 to C_3 such that $X \subseteq \operatorname{dom} f$ and f is a constant on X holds f is continuous on X.

(147) For every partial function f from C_1 to R_1 such that $X \subseteq \operatorname{dom} f$ and f is a constant on X holds f is continuous on X.

(148) For every partial function f from R_1 to C_1 such that $X \subseteq \operatorname{dom} f$ and f is a constant on X holds f is continuous on X.

(149) Let f be a partial function from C_1 to C_1. Suppose that for every point x_0 of C_1 such that $x_0 \in \operatorname{dom} f$ holds $f_{x_0} = x_0$. Then f is continuous on $\operatorname{dom} f$.

(150) For every partial function f from C_1 to C_1 such that $f = \mathrm{id}_{\operatorname{dom} f}$ holds f is continuous on $\operatorname{dom} f$.

(151) Let f be a partial function from C_1 to C_1 and Y be a subset of C_1. If $Y \subseteq \operatorname{dom} f$ and $f \restriction Y = \mathrm{id}_Y$, then f is continuous on Y.

(152) Let f be a partial function from C_1 to C_1, z be a complex number, and p be a point of C_1. Suppose $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f_{x_0} = z \cdot x_0 + p$. Then f is continuous on X.

(153) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Suppose that for every point x_0 of C_1 such that $x_0 \in \operatorname{dom} f$ holds $f_{x_0} = \|x_0\|$. Then f is continuous on $\operatorname{dom} f$.

(154) Let f be a partial function from the carrier of C_1 to $\mathbb{R}$. Suppose $X \subseteq \operatorname{dom} f$ and for every point x_0 of C_1 such that $x_0 \in X$ holds $f_{x_0} = \|x_0\|$. Then f is continuous on X.

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